Analysis III - 203(d)

Winter Semester 2024

Session 14: December 19, 2024

Exercise 1 *Given the following functions over an interval* $[0, 1)$*,*

(a) $f(x) = x$

(b) $g(x) = x^2$

(c) $h(x) = e^x$

(d) $s(x) = sin(\pi x)$

sketch their extension to

- *a periodic function with period* 1*,*
- *an even periodic function with period* 2*,*
- *an odd periodic function with period* 2*.*

State the formulas for the even and odd 2*-periodic extensions over the interval* $[-1, 1]$*.*

Exercise 2 *Consider the function*

$$f : [0, 1] \to \mathbb{R}, \quad x \mapsto x^3.$$

Extend this to an odd function with period $T = 2$*. Sketch the graph of that function from* -2 *to* 2*. Compute its Fourier coefficients in standard form. Compute the complex Fourier coefficients.*

Exercise 3 *Suppose that*

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0 \ , \\ 1-x & \text{if } 0 < x < 1 \ , \\ 0 & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1 \ , \\ 0 & \text{otherwise.} \end{cases} \qquad h(x) = |x|.$$

Compute the convolutions $u(x) = (f \star g)(x)$ *and* $v(x) = (g \star g)(x)$ *and* $w(x) = (g \star h)(x)$*.*

Exercise 4 *Suppose that* $f(x) = x^2$ *and that*

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1 \ , \\ 0 & \text{otherwise.} \end{cases} \qquad h(x) = \begin{cases} e^{-x} & \text{if } x > 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

Compute the convolutions $u(x) = (f \star g)(x)$ *and* $v(x) = (f \star h)(x)$ *.*

Exercise 5 *We have discussed solutions to the differential equation*

$$-\Delta u(x) + k^2 u(x) = e^{-|x|}, \quad x \in \mathbb{R}.$$

- *Verify that, in the case $k = 1$, we have a solution*

  $$u(x) = \frac{1}{2}(1 + |x|)e^{-|x|}$$

  Verify that every function of the form

  $$v(x) = \frac{1}{2}(1 + |x|)e^{-|x|} + c_1 e^{-x} + c_2 e^{x}$$

  is a solution. For which values of c_1 and c_2 does the function decay towards zero as x goes to $\pm\infty$?

- *Verify that, in the case $k \neq 1$, we have a solution*

  $$u(x) = -\frac{e^{-k|x|}}{k(k^2 - 1)} + \frac{e^{-|x|}}{k^2 - 1}$$

  Verify that every function of the form

  $$v(x) = -\frac{e^{-k|x|}}{k(k^2 - 1)} + \frac{e^{-|x|}}{k^2 - 1} + c_1 e^{-kx} + c_2 e^{kx}$$

  is a solution.

Exercise 6 *We want to find a solution to the boundary value problem*

$$\begin{aligned} -\Delta u(x) + k^2 u(x) &= x, \quad 0 < x < L, \\ u(0) = 0, &\quad u(L) = 0. \end{aligned}$$

- *Extend the right-hand side $f(x) = x$ to an odd function with period $2L$ and compute its Fourier coefficients.*

- *Using these coefficients, find the Fourier series of the solution u. Verify that the boundary condition $u(0) = u(L) = 0$ is satisfied.*

Exercise 7 (Fun with Neumann boundary conditions) *Consider the Poisson problem with Neumann boundary conditions over the interval $[a, b] = [0, 1]$:*

$$\begin{aligned} -u''(x) + k^2 u(x) &= x - \frac{1}{2}, \quad a < x < b, \\ u'(a) = 0, &\quad u'(b) = 0, \end{aligned}$$

for some $k \geq 0$.

(a) *Extend* $f(x) = x - \frac{1}{2}$ *to an even function over the real line with period* 2.

(b) *Compute the Fourier coefficients of that even extension of* f.

(c) *Find the Fourier series of a function that satisfies the above differential equation.*

Exercise 8 (Fun with periodic boundary conditions) *Consider the Poisson problem with periodic boundary conditions over the interval* $[a, b] = [0, 1]$*:*

$$\begin{gathered} -u''(x) + k^2 u(x) = x - \frac{1}{2}, \quad a < x < b, \\ u(a) = u(b), \end{gathered}$$

for some $k \geq 0$.

(a) *Extend* $f(x) = x - \frac{1}{2}$ *to a* 1*-periodic function over the real line.*

(b) *Compute the Fourier coefficients of that extension of* f.

(c) *Find the Fourier series of a function that satisfies the above differential equation.*