Analysis III - 203(d)

Winter Semester 2024

Session 13: December 12, 2024

Exercise 1 *The Fourier transform of*

$$f(x) = e^{-5x^2}$$

is the function

$$\hat{f}(\alpha) = \frac{1}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}.$$

Find the Fourier transforms of

$$f', \quad f'', \quad f''', \quad f'''', \quad g(x) = f(2x), \quad h(x) = f(x-3).$$

Exercise 2 *Find* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$\hat{f}(\alpha) = \frac{3}{1+\alpha^2} + \frac{-1}{1+4\alpha^2} + \frac{\sin(4\alpha+3)}{4\alpha+3}$$

Exercise 3 *We consider the Poisson problem with Dirichlet boundary conditions over the interval* $[0, L]$*:*

$$-\Delta u(x) = x^2, \quad 0 < x < L,$$
$$u(0) = 1, \quad u(L) = 2$$

- *Solve this problem directly. The solution is a polynomial of order* 4.
- *Extend the right-hand side* $f(x) = x^2$ *to an odd function with period* $2L$ *and compute its Fourier coefficients.*
- *Using these coefficients, find the Fourier series of the solution* u^f*, which solves*

  $$-\Delta u^f(x) = x^2, \quad 0 < x < L,$$
  $$u^f(0) = 0, \quad u^f(L) = 0.$$

  How do you use the superposition principle to solve the full problem?

Exercise 4 *Directly compute the solution of the problem*

$$-\Delta u(x) = x^2, \quad 0 < x < L,$$
$$u(0) = 0, \quad u(L) = 0,$$

using elementary analysis. Then find the Fourier coefficients of its odd extension to the interval $[-L, L]$*. Compare this with the function* u^f *from the previous exercise.*

Exercise 5 *We solve the Poisson problem with homogeneous Dirichlet boundary conditions over* $[0,1]$*:*

$$-\Delta u(x) = \begin{cases} x & \text{if } 0 < x \leq 0.5 \\ 0 & \text{if } 0.5 < x \leq 1 \end{cases}, \qquad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0$$

- *Extend the right-hand side* $f(x)$ *to an odd function with period* 2 *and compute its Fourier coefficients.*
- *Using these coefficients, find the solution* u*. Verify that the boundary condition* $u(0) = u(1) = 0$ *is satisfied.*

Exercise 6 *Consider the following Fourier sine series for two functions* $u, f : \mathbb{R} \to \mathbb{R}$ *with period* T*:*

$$u(x) = \sum_{n=1}^{\infty} b_n^u \sin\left(\frac{2\pi n}{T} x\right), \quad f(x) = \sum_{n=1}^{\infty} b_n^f \sin\left(\frac{2\pi n}{T} x\right).$$

Express the Fourier coefficients of f *by the Fourier coefficients of* u*, and vice-versa, express the Fourier coefficients of* u *by the Fourier coefficients of* f*, if* u *and* f *are related by the following differential equations:*

(a) $-u''(x) = f(x)$.

(b) $u''''(x) = f(x)$.

(c) $-u''(x) + 25u(x) = f(x)$.

(d) $u''''(x) - u''(x) = f(x)$.

(e) $-u''(x) + \gamma \cdot u''(x) = f(x)$.

In the last item, $\gamma \in \mathbb{R}$ *is a constant. For which values of* γ *can you always find a solution?*

Exercise 7 *Recall the definition of the Fourier transform:*

$$\mathfrak{F}[f](\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-it\alpha} \, dt \tag{1}$$

Let $a, b, c \in \mathbb{R}$ *be real parameters with* $a \neq 0$*. Compute the Fourier transforms of the following:*

$$g(t) = f(at), \tag{2}$$
$$h(t) = e^{-itb} f(t), \tag{3}$$
$$m(t) = f(t - c). \tag{4}$$

Hint: you have the first two in the lecture and in textbook. You can compute them using results from the lecture or via some standard integral manipulations.