Analysis III - 203(d)

Winter Semester 2024

Session 12: December 5, 2024

Exercise 1 *Consider the following functions with period T:*

- *A function f with period $T = 1$ such that*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 0.5 \\ 0 & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

- *A function g with period $T = 2\pi$ such that*

$$g(x) = \begin{cases} x & \text{if } 0 \leq x < \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

- *A function h with period $T = 1$ such that*

$$h(x) = -x \text{ if } 0 \leq x < 1.$$

Find the Fourier coefficients of the Fourier series of these functions. You can use the Fourier series seen in the lecture to get the coefficients.

Exercise 2 *Explicitly write down the coefficients a_n and b_n and the periods of the following Fourier series:*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \sin(2\pi(2n+1)x),$$
$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos(2\pi(2n-1)x),$$
$$h(x) = \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(\pi n x).$$

Determine the Fourier coefficients of the derivatives of those functions.

Exercise 3 *Let $f(x)$ be a periodic function with period T, represented by its Fourier series:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right) \right).$$

As explained in the lecture,

$$\int_{x_0}^{x} f(x)dx = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} a_n \int_{x_0}^{x} \cos\left(\frac{2\pi nx}{T}\right) dx + b_n \int_{x_0}^{x} \sin\left(\frac{2\pi nx}{T}\right) dx.$$

Complete this discussion and find the Fourier series of a function $g(x)$ such that

$$\int_{x_0}^{x} f(x) = cx + g(x) \tag{1}$$

for some $c \in \mathbb{R}$.

Exercise 4 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions. Recall that a function is called even if*

$$f(-x) = f(x)$$

and odd if

$$f(-x) = -f(x)$$

- *Show that if f and g are both odd or both even, then fg is even*
- *Show that if one of f and g is odd and the other is even, then fg is odd.*
- *Show that the only function that is both odd and even has constant value zero.*

Exercise 5 *Compute the Fourier transform of the function*

$$f(x) = \begin{cases} x & \textit{if } 0 \le x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

You can either directly use the complex exponential, or you can express it in terms of the sine and cosine function.
(Interpretation: the function $f(x)$ describes a localized signal: it is zero at $x = 0$, then it rises linearly up to 1, and then it jumps back to zero and remains zero from there on. The signal is not periodic.)

Exercise 6 *Find the Fourier transform of*

$$f(x) = \begin{cases} \sin(x) & \textit{if } 0 \le x \le 2\pi \\ 0 & \textit{otherwise} \end{cases}$$

Exercise 7 (Extra) *We have introduced the Fourier transform*

$$\mathfrak{F}(f)(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

Different authors define the Fourier transform alternatively by:

$$\mathfrak{F}_2(f)(\xi) := \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx, \quad \mathfrak{F}_3(f)(\omega) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Express $\mathfrak{F}_2(f)$ and $\mathfrak{F}_3(f)$ in terms of $\mathfrak{F}(f)$.