Analysis III - 203(d)

Winter Semester 2024

Session 8: November 7, 2024

Exercise 1 *Consider the surface*

$$S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \|x\| = 1, x_3 > 0 \right\}$$

- *Find a parameterization of S and find the unit normal corresponding to that parameterization.*
- *Find a parameterization of $C = \partial S$.*
- *Verify Stokes theorem with the vector field*

$$\vec{F}(x_1, x_2, x_3) = (-x_2, x_1, x_3).$$

Exercise 2 *Let S be the surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 = 1,\ -1 < x_3 < 1 \right\}, \tag{1}$$

Suppose we have a vector field

$$\vec{F}(x_1, x_2, x_3) := (x_2 x_3, -x_1 x_3, 0). \tag{2}$$

- *Find a parameterization of the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar field $\vec{F} \cdot \vec{n}$ and calculate the integrals of this scalar fields over S.*
- *Compute the integral using the formula*

$$\int_\Omega \vec{F} \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt \tag{3}$$

  and compare the result.
- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*
- *Compute the integrals*

$$\oint_C \vec{F} dl. \tag{4}$$

Exercise 3 *Let S be a surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 > 0 \right\}, \tag{5}$$

$$\Omega = \left\{ (s,t) \in \mathbb{R}^2 \;\middle|\; s + t < 1,\ s, t > 0 \right\}, \tag{6}$$

$$\Phi : \Omega \to S, \quad (s,t) \mapsto (s, t, 1 - s - t). \tag{7}$$

We consider the vector fields

$$\vec{F}(x_1, x_2, x_3) := (x_2 x_3, -x_1 x_3, x_1 x_2), \quad \vec{G}(x_1, x_2, x_3) := (-x_2, x_1, x_3) \tag{8}$$

- *Find the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar fields $\vec{F} \cdot \vec{n}$ and $\vec{G} \cdot \vec{n}$ and calculate the integrals of these scalar fields over S.*
- *Compute the integrals using the formula*

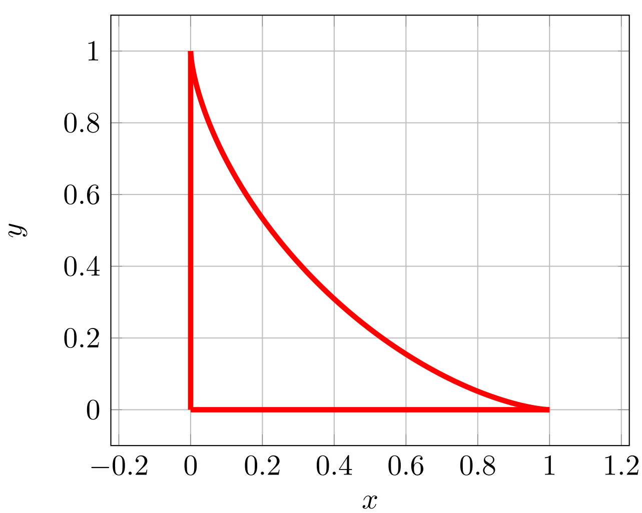
$$\int_\Omega \vec{F} \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt \tag{9}$$

  and compare the result.
- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*
- *Compute the integrals*

$$\oint_C \vec{F} dl, \quad \oint_C \vec{G} dl. \tag{10}$$

- *Can you exclude that $\vec{F}$ or $\vec{G}$ are gradients of a scalar field?*

Exercise 4 (Green's theorem) *Consider the following closed curve C:*



The curved part of the curve can be parameterized by

$$\gamma : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad t \mapsto \left(\cos^3(t), \sin^3(t)\right).$$

- *Find parameterizations of the other two curves.*
- *With the* $\vec{F}(x_1, x_2) = (x_2, -x_1)$ *compute the curve integral*

$$\oint_C \vec{F} dl$$

- *Use Green's theorem to explain how to compute the size of the area enclosed by the curve* C*.*

Exercise 5 (Stokes's theorem) *We have a vector field*

$$\vec{F}(x_1, x_2, x_3) = (x_1 x_2, x_2 x_3, x_1 x_3)$$

and a surface

$$S := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1,\ x_2 \geq 0, x_3 \geq 0\right\}$$

- *Find parameterizations of the surface* S *and a parameterization of its boundary curve* C*.*
- *Use Stokes' theorem to compute the integral*

$$\iint_S \operatorname{curl} \vec{F} \, d\sigma.$$