Analysis III - 203(d)

Winter Semester 2024

Session 7: October 31, 2024

Exercise 1 *Verify the divergence theorem for the following vector field $\vec{F}$ and volume V:*

$$\vec{F}(x_1, x_2, x_3) := (x_1 x_3, x_2, x_2), \qquad V := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 < 1 \right\}.$$

Note that V is just the three-dimensional unit ball.

Exercise 2 *Consider the volume*

$$V := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 < 5 \right\}, \tag{1}$$

- *What is the surface S of this volume?*
- *Find the outward pointing unit normal $\vec{n}$ along the surface S of this volume. Write $\vec{n}$ in terms of (x_1, x_2, x_3) at any point on the surface S.*
- *Find a parameterization of the surface S.*
- *Find a vector field $\vec{F}$ such that $\vec{F} \cdot \vec{n} = x_1^2 + x_2 + x_3$ along the surface S.*
- *Use the divergence theorem to compute*

$$\iint_S x_1^2 + x_2 + x_3 dx_1 dx_2 dx_3. \tag{2}$$

Exercise 3 *Consider the volume*

$$V := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 < x_3 < 1 \right\}$$

Find the the boundary S of this volume and compute its surface area.

Exercise 4 *Find a regular parameterization $\Phi(s,t)$ of the surface*

$$S := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; 0 < x_3 < 1,\ x_1^2 + x_2^2 = 1 + x_3^2 \right\}$$

Compute cross product $\partial_s \Phi(s,t) \times \partial_t \Phi(s,t)$ and its norm.

Exercise 5 *Consider the volume*

$$V := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 < 1,\ x_1, x_2, x_3 > 0 \right\}, \tag{3}$$

Suppose we have a vector field

$$\vec{F}(x_1, x_2, x_3) = \left(x_1^2 x_2, 3x_2^2 x_3, 9x_3^2 x_1 \right) \tag{4}$$

Use the divergence theorem to compute the surface integral

$$\iint_S \vec{F} d\sigma. \tag{5}$$

Exercise 6 *Given the curve*

$$\gamma : [0, \pi] \to \mathbb{R}^2, \quad t \mapsto (3\cos(2t), \sin(2t))$$

and a function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto \left(x_1^2 + 81 x_2^2 \right)^{\frac{3}{2}},$$

compute the integrals

$$\int_\Gamma f\ d\ell, \quad \int_\Gamma \nabla f\ d\ell.$$