Analysis III - 203(d)

Winter Semester 2024

Session 5: October 10, 2024

Exercise 1 *Consider a curve u in one-dimensional space with*

$$u : [1,2] \to \mathbb{R}, \qquad t \mapsto t^2 + t$$

Verify that the curve is simple, differentiable, and regular. Compute the curve integral $\int_u f \; dl$, where

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \to 3x^3$$

is the scalar field.

Exercise 2 (vector analysis in 1D) *Let $\Omega \subseteq (a,b)$ be an open interval in one-dimensional space.*

- *Explain why there cannot be a simple closed continuous curve in Ω.*
- *When $\Omega = (-10, 10)$, compute the integral of the scalar field*

  $$f(x) = \frac{x}{\sqrt{1+x^2}}$$

  along the curves

  $$\begin{gathered} \gamma_1 : [0,1] \to \Omega, \quad t \mapsto (2t-1), \\ \gamma_2 : [-1,1] \to \Omega, \quad t \mapsto (t), \\ \gamma_3 : [0,1] \to \Omega, \quad t \mapsto (1-2t), \\ \gamma_4 : [0,1] \to \Omega, \quad t \mapsto (-1+2t^5), \end{gathered}$$

  Compute the tangent vectors $\dot{\gamma}(t)$.
- *Compute the integral of the vector field*

  $$F(x) = \left(x e^{x^2} \right) \tag{1}$$

  along the curve γ_4. Find a potential for this vector field, and write down the general form of all potentials.

Exercise 3 *We review notions of potentials and conservative vector fields. Let $\Omega \subseteq \mathbb{R}^n$ be open. Suppose we have a vector field $F = (F_1, \ldots, F_n) \in C^1(\Omega, \mathbb{R}^n)$. Recall that we have introduced the condition*

$$\partial_i F_j = \partial_j F_i, \qquad 1 \leq i, j \leq n. \tag{2}$$

- *Suppose that $n = 2$. Show that F satisfies (2) if and only if it is curl-free:* $\operatorname{curl} F = 0$.
- *Suppose that $n = 3$. Show that F satisfies (2) if and only if it is curl-free:* $\operatorname{curl} F = 0$.
- *Suppose that $n = 1$. Show that F satisfies (2).*
- *Suppose that F admits a potential $f \in C^1(\Omega, \mathbb{R})$, so that $\nabla f = F$. Show that if $\gamma : [a, b] \to \Omega$ is a simple regular curve, then*

  $$\int_\gamma F \, dl = f(\gamma(b)) - f(\gamma(a)). \tag{3}$$

  Show that if γ is closed, then

  $$\int_\gamma F \, dl = 0. \tag{4}$$

Exercise 4 *We introduce the following curves:*

$$\gamma : [0, 1] \to \mathbb{R}^3, \qquad t \mapsto \left(3, t^2, 4t\right),$$
$$\delta : [1, \infty) \to \mathbb{R}^2, \qquad t \mapsto \left(5, e^{-t}\right)$$

For each curve

- *compute the tangent vector*
- *compute the speed of the curve*
- *find the unit tangent vector*
- *for δ, find the unit normal along the curve that is the 90 degree clockwise rotation of unit tangent*
- *argue why it is a regular curve*
- *and compute the length of the curve.*

Exercise 5 *We consider the vector field*

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto \left(x^3, y^3\right)$$

We want to find a potential over the domain $\Omega = \mathbb{R}^2$. Fix a constant of integration at $(0, 0)$ and define a potential via the integral of the vector field F along a simple regular curve going from $(0, 0)$ to (x, y).

Exercise 6 *The closed curve*

$$\gamma(t) = (\sin(t)(1 + 0.5\sin(2t)), \cos(t)(1 + 0.5\sin(2t)))$$

encircles a domain Ω *in counterclockwise direction. Find the tangent* $\dot{\gamma}(t)$*, the unit tangent* $\tau(t)$ *and the outward pointing unit normal* $\vec{n}(t)$*. Only simplify as much as reasonable.*

Exercise 7 *We work over the quadratic domain*

$$\Omega := \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, -1 < x_1 < 1,\ -1 < x_2 < 1\right\}.$$

Compute the integral $\iint_\Omega \mathrm{div}\vec{F}\; dx_1 dx_2$*, where*

$$\vec{F}(x_1, x_2) = \left(\sin(x_1)x_2, \left(x_1^2 + x_2\right)^5\right)$$