

Analysis III - 203(d)

Winter Semester 2024

Session 4: October 3, 2024

Exercise 1 Compute the line integral of the vector field $\vec{F}$ along the curve γ , where

$$\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x_1, x_2) \mapsto (x_2, 0), \quad \gamma : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$$

Compute the line integral of the vector field $\vec{G}$ along the curve δ , where

$$\vec{G} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (2, 3, -1), \quad \delta : [0, 4] \rightarrow \mathbb{R}^3, \quad t \mapsto (t^2, \cos(t), e^t)$$

Solution 1

$$\begin{aligned} \int_{\Gamma} \vec{F}(x_1, x_2) d\ell &= \int_0^{2\pi} (\vec{F} \circ \gamma)(t) \cdot \dot{\gamma}(t) dt \\ &= \int_0^{2\pi} \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt \\ &= \int_0^{2\pi} -\sin^2(t) dt \\ &= \int_0^{2\pi} -\frac{1}{2} + \frac{1}{2} \cos(2t) dt = \left[-\frac{1}{2}t + \frac{1}{4} \sin(2t) \right]_0^{2\pi} = -\pi \end{aligned}$$

$$\begin{aligned} \int_{\Gamma} \vec{G}(x_1, x_2, x_3) d\ell &= \int_0^4 (\vec{G} \circ \delta)(t) \cdot \dot{\delta}(t) dt \\ &= \int_0^4 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ -\sin(t) \\ e^t \end{pmatrix} dt \\ &= \int_0^4 4t - 3\sin(t) - e^t dt \\ &= [2t^2 + 3\cos(t) - e^t]_0^4 \\ &= 32 + 3\cos 4 - e^4 - (3 - 1) = 30 + 3\cos 4 - e^4 \end{aligned}$$

Exercise 2 Compute the line integral of the vector field $\vec{F}$ along the curve γ , where

$$\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x_1, x_2) \mapsto (e^{x_1 x_2} x_2, e^{x_1 x_2} x_1)$$

and

$$\gamma : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto \left(\arctan(\cos(\pi t)^2 - \sin(\pi t)^2), 1 + \sqrt[2]{1 + \arctan(t)} \right)$$

Solution 2 The vector field is the gradient of $e^{x_1 x_2}$, and so we can use the formula for vector fields that admit a potential. We find that

$$\int_{\Gamma} \vec{F} d\ell = \int_{\Gamma} \nabla f d\ell = f(\gamma(1)) - f(\gamma(0))$$

Now we only need to compute the function values of f . Explicitly,

$$\begin{aligned} \gamma(0) &= \left(\arctan(\cos(\pi 0)^2 - \sin(\pi 0)^2), 1 + \sqrt[2]{1 + \arctan(0)} \right) \\ &= \left(\arctan(1), 1 + \sqrt[2]{1} \right) = (\pi/4, 2) \end{aligned}$$

$$\begin{aligned} \gamma(1) &= \left(\arctan(\cos(\pi 1)^2 - \sin(\pi 1)^2), 1 + \sqrt[2]{1 + \arctan(1)} \right) \\ &= \left(\arctan(1), 1 + \sqrt[2]{1 + \pi/4} \right) = \left(\pi/4, 1 + \sqrt[2]{1 + \pi/4} \right) \end{aligned}$$

We then compute

$$f(\gamma(1)) - f(\gamma(0)) = \exp\left(\frac{\pi}{4}(1 + \sqrt[2]{1 + \pi/4})\right) - \exp(\pi/2).$$

Exercise 3 What is the length of the graph of the function $g(x) = x^{\frac{3}{2}}$ over the interval $[0, 2]$? Simplify as much as reasonable.

Solution 3 We can express the graph as a curve

$$\gamma : [0, 2] \rightarrow \mathbb{R}^2, \quad t \mapsto (t, t^{\frac{3}{2}})$$

As mentioned in the lecture, the integral of 1 along that curve gives the length. Hence we find:

$$\int_{\Gamma} 1 d\ell = \int_0^2 \left| \left(1, \frac{3}{2} t^{\frac{1}{2}} \right) \right| dt = \int_0^2 \sqrt{1 + \frac{9t}{4}} dt$$

We integrate the last expression. The integrand is a derivative, up to some numerical factor:

$$\begin{aligned} \int_0^2 \sqrt{1 + \frac{9}{4}t} dt &= \frac{2}{3} \cdot \frac{4}{9} \int_0^2 \frac{3}{2} \cdot \frac{9}{4} \left(1 + \frac{9}{4}t \right)^{\frac{1}{2}} dt \\ &= \frac{2}{3} \cdot \frac{4}{9} \int_0^2 \partial_t \left(1 + \frac{9}{4}t \right)^{\frac{3}{2}} dt \\ &= \frac{2}{3} \cdot \frac{4}{9} \left[\left(1 + \frac{9}{4}t \right)^{\frac{3}{2}} \right]_0^2 \\ &= \frac{2}{3} \cdot \frac{4}{9} \left(\left(1 + \frac{9}{4} \right)^{\frac{3}{2}} - 1 \right) \end{aligned}$$

$$= \frac{8}{27} \left(\left(\frac{11}{2} \right)^{\frac{3}{2}} - 1 \right).$$

(You can compute the integral via *Intégration par changement de variable*, which is the same calculation but in a different formalism.)

Exercise 4 Take a look at the functions

$$f(x_1, x_2) = 1, \quad g(x_1, x_2) = x_2.$$

We have the following curves:

$$\begin{aligned} \alpha : [0, 1] &\rightarrow \mathbb{R}^2, & t &\mapsto (t, t), \\ \beta : [0, 2] &\rightarrow \mathbb{R}^2, & t &\mapsto \begin{cases} (t, 0) & 0 \leq t < 1, \\ (0, t-1) & 1 \leq t \leq 2, \end{cases} \\ \gamma : [0, 1] &\rightarrow \mathbb{R}^2, & t &\mapsto (t^2, t). \end{aligned}$$

Compute:

$$\int_A f \, d\ell, \quad \int_B f \, d\ell, \quad \int_A g \, d\ell, \quad \int_B g \, d\ell, \quad \int_\Gamma g \, d\ell.$$

Solution 4 We compute these integrals as follows:

$$\int_A f \, d\ell = \int_0^1 \sqrt{2} \, dt = \left[\sqrt{2}t \right]_0^1 = \sqrt{2}, \quad \int_B f \, d\ell = \int_0^2 dt = [t]_0^2 = 2$$

$$\int_A g \, d\ell = \int_0^1 t\sqrt{2} \, dt = \left[\frac{\sqrt{2}}{2} t^2 \right]_0^1 = \frac{\sqrt{2}}{2}$$

$$\int_B g \, d\ell = \int_0^1 (0) \, dt + \int_1^2 (t-1) \, dt = \int_1^2 (t-1) \, dt = \left[\frac{1}{2}(t-1)^2 \right]_1^2 = \frac{1}{2}$$

For the integral of g over the third curve, we proceed as follows:

$$\begin{aligned} \int_\Gamma g \, d\ell &= \int_0^1 t\sqrt{4t^2+1} \, dt \\ &= \int_0^1 t(4t^2+1)^{\frac{1}{2}} \, dt \\ &= \frac{2}{3} \cdot \frac{1}{8} \cdot \int_0^1 \frac{3}{2} \cdot 8t(4t^2+1)^{\frac{1}{2}} \, dt \\ &= \frac{2}{3} \cdot \frac{1}{8} \cdot \int_0^1 \partial_t (4t^2+1)^{\frac{3}{2}} \, dt = \frac{2}{3} \cdot \frac{1}{8} \cdot \left(5^{\frac{3}{2}} - 1 \right) = \frac{5^{\frac{3}{2}} - 1}{12}. \end{aligned}$$

Note that the curves all have the same starting and end points, $(0,0)$ and $(1,1)$, but the curve integrals of the function may (generally) differ from curve to curve.

Exercise 5 We want to verify that the curve integrals do not depend on the direction of the parameterization. Consider the curve parameterizations

$$\gamma_+ : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (t, 1 + \frac{1}{2}t^2), \quad (1)$$

$$\gamma_- : [0, 1] \rightarrow \mathbb{R}^2, \quad s \mapsto (1 - s, 1 + \frac{1}{2}(1 - s)^2), \quad (2)$$

- Given the scalar field

$$f(x, y) = x(y - \frac{1}{2}x^2) = xy - \frac{1}{2}x^3,$$

show that

$$\int_{\gamma_+} f \, dl = \int_{\gamma_-} f \, dl \quad (3)$$

- Compute the tangent and unit tangent vectors of each curve, and show that $\dot{\gamma}_+(t) = -\dot{\gamma}_-(1 - t)$. Show that the curve integrals along γ_+ and γ_- of the vector field

$$F(x, y) = (-y, x) \quad (4)$$

are the negative of each other.

Solution 5 • We compute the first integral directly:

$$\begin{aligned} \int_{\gamma_+} f \, dl &= \int_0^1 f(t, 1 + \frac{1}{2}t^2) |\dot{\gamma}_+(t)| dt \\ &= \int_0^1 t \sqrt{1 + t^2} dt \\ &= \frac{1}{2} \int_0^1 2t \sqrt{1 + t^2} dt. \end{aligned}$$

We substitute $u = 1 + t^2$, so that $du = 2dt$. Thus

$$\frac{1}{2} \int_0^1 2t \sqrt{1 + t^2} dt = \frac{1}{2} \int_1^2 \sqrt{u} \, du = \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^2 = \frac{1}{3} (2\sqrt{2} - 1).$$

The second integral can be computed directly as well, or with a change of variables. For the latter, we use

$$\begin{aligned} \int_{\gamma_-} f \, dl &= \int_0^1 f(1 - s, 1 + \frac{1}{2}(1 - s)^2) |\dot{\gamma}_-(s)| ds \\ &= - \int_1^0 f(t, 1 + \frac{1}{2}t^2) |\dot{\gamma}_-(1 - t)| dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 t \sqrt{1 + (1 - (1 - t))^2} dt = \int_0^1 t \sqrt{1 + t^2} dt \\
&= \int_{\gamma_+} f dl,
\end{aligned}$$

where we used the substitution $t = 1 - s$ with $dt = -ds$.

- Given the two parameterizations as stated above,

$$\gamma_+ : [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto (t, 1 + \frac{1}{2}t^2), \quad (5)$$

$$\gamma_- : [0, 1] \rightarrow \mathbb{R}^2, \quad s \mapsto (1 - s, 1 + \frac{1}{2}(1 - s)^2), \quad (6)$$

We compute the tangent vectors:

$$\dot{\gamma}_+(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad \dot{\gamma}_-(s) = \begin{pmatrix} -1 \\ s - 1 \end{pmatrix}.$$

Their norms are

$$\begin{aligned}
|\dot{\gamma}_+(t)| &= \sqrt{1 + t^2}, \\
|\dot{\gamma}_-(s)| &= \sqrt{1 + (s - 1)^2}.
\end{aligned}$$

It is intuitive that $\dot{\gamma}_+(t) = -\dot{\gamma}_-(1 - t)$, as we check easily:

$$\dot{\gamma}_-(1 - t) = - \begin{pmatrix} -1 \\ 1 - t - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix} = \dot{\gamma}_+(t).$$

Finally, for the curve integral we find again, using the change of variables $s = 1 - t$, hence $ds = -dt$, the following identity:

$$\begin{aligned}
\int_{\gamma_+} F dl &= \int_0^1 F(\gamma_+(t)) \cdot \dot{\gamma}_+(t) dt \\
&= - \int_0^1 F(\gamma_-(1 - t)) \cdot \dot{\gamma}_-(1 - t) dt \\
&= \int_1^0 F(\gamma_-(s)) \cdot \dot{\gamma}_-(s) ds.
\end{aligned}$$

We switch the bounds of integration into normal order and find

$$\int_1^0 F(\gamma_-(s)) \cdot \dot{\gamma}_-(s) ds = - \int_0^1 F(\gamma_-(s)) \cdot \dot{\gamma}_-(s) ds = - \int_{\gamma_-} F dl.$$

Exercise 6 Consider the open cube

$$\Omega = (0, 1)^2 = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\}.$$

- Show that Ω is convex.
- Let $z \in \Omega$. Show that Ω is star-shaped with respect to z .
- Show that Ω is simply-connected.

For the last part, show that any two continuous curves with the same endpoints are homotopic relative to endpoints.

Solution 6

Graphically, it is clear that each segment between points of Ω is also contained in Ω . But we prove this formally.

Suppose that $a = (a_1, a_2), b = (b_1, b_2) \in \Omega$ and $t \in [0, 1]$. We want to show that $ta + (1 - t)b \in \Omega$. We compute that

$$\begin{aligned} ta + (1 - t)b &= (ta_1 + (1 - t)a_2, tb_1 + (1 - t)b_2) \\ &= (ta_1 + (1 - t)a_2, tb_1 + (1 - t)b_2). \end{aligned}$$

This point is in Ω if each coordinate is in the interval $(0, 1)$. We show that for the first coordinate, because the argument for the second coordinate is almost the same. Clearly,

$$ta_1 + (1 - t)a_2 > t \min(a_1, a_2) + (1 - t) \min(a_1, a_2) = \min(a_1, a_2) > 0,$$

$$ta_1 + (1 - t)a_2 < t \max(a_1, a_2) + (1 - t) \max(a_1, a_2) = \max(a_1, a_2) < 1.$$

We conclude that Ω is convex.

As already observed, the segment between any two points $z, a \in \Omega$ is already contained in Ω . If we fix any point $z \in \Omega$, then Ω is star-shaped. Note that we do not use any property except that Ω is convex. Indeed, every convex domain is also star-shaped (with respect to any point inside).

Given two continuous curves $\gamma_0, \gamma_1 : [a, b] \rightarrow \Omega$ with $\gamma_0(a) = \gamma_1(a) =: g_a$ and $\gamma_0(b) = \gamma_1(b) =: g_b$, we define

$$\gamma : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$$

by setting

$$\gamma(t, s) = (1 - s) \cdot \gamma_0(t) + s \cdot \gamma_1(t).$$

For any $t \in [a, b]$ we have $\gamma_0(t), \gamma_1(t) \in \Omega$, and since Ω is convex, we have $(1 - s) \cdot \gamma_0(t) + s \cdot \gamma_1(t) \in \Omega$ for all $s \in [0, 1]$. Hence γ defines a function

$$\gamma : [a, b] \times [0, 1] \rightarrow \Omega$$

Furthermore, γ is obviously continuous in both variables.

We verify the remaining conditions. For all $t \in [a, b]$ we have

$$\gamma(t, 0) = (1 - 0) \cdot \gamma_0(t) + 0 \cdot \gamma_1(t) = \gamma_0(t) \quad (7)$$

and

$$\gamma(t, 1) = (1 - 1) \cdot \gamma_0(t) + 1 \cdot \gamma_1(t) = \gamma_1(t). \quad (8)$$

Additionally, for all $s \in [0, 1]$, we have

$$\gamma(a, s) = (1 - s) \cdot \gamma_0(a) + s \cdot \gamma_1(a) = (1 - s) \cdot g_a + s \cdot g_a = g_a \quad (9)$$

and

$$\gamma(b, s) = (1 - s) \cdot \gamma_0(b) + s \cdot \gamma_1(b) = (1 - s) \cdot g_b + s \cdot g_b = g_b. \quad (10)$$

This shows that γ satisfies the required axioms.