Analysis III - 203(d)

Winter Semester 2024

Session 3: September 26, 2024

Exercise 1 *Consider the scalar field* $f(x_1, x_2) = x_1 x_2$.

- *Describe its level sets for the values* $-2, -1, 0, 1, 2$.
- *Draw the gradient at a few points, such as*

$$(1,0)\,,(0,1)\,,(1,1)\,,(-1,-1)\,,\left(\sqrt{2},\sqrt{2}\right),\left(\sqrt{2},-\sqrt{2}\right) \tag{1}$$

- *Recall the hyperbolic sine and hyperbolic cosine functions* $\sinh(t)$ *and* $\cosh(t)$. *Sketch the curve*

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2, \quad t \mapsto (\ \cosh(t),\ \sinh(t)\ ) \tag{2}$$

  and compute its tangent vector $\dot{\gamma}$.

Exercise 2 *Consider the curves*

$$\gamma : [-1,1] \to \mathbb{R}^2, \quad t \mapsto (t^3, t^2)$$

$$\delta : [-1,1] \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(2t))$$

Draw them. Are they simple, closed, differentiable, or regular? What are their tangents? Show that γ *is the graph of a function in the first coordinate.*

Exercise 3 *Compute the line integral of*

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto 1 + x_1^2 + x_2^2$$

along the two curves γ *and* δ, *given by*

$$\gamma : [0,\pi] \to \mathbb{R}^2, \quad t \mapsto (-\cos(t), \sin(t))$$

$$\delta : [-1,1] \to \mathbb{R}^2, \quad t \mapsto (t, 0)$$

Compare the results. What are the endpoints of the two curves?

Exercise 4 *Compute the line integral of* $f : \mathbb{R}^2 \to \mathbb{R}$ *along the circle* C *with radius* 3 *centered at the origin, where*

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto 3x_2^2 + x_2^3$$

Hint: you must first find a parameterization of C.

Exercise 5 *You are excavating a tunnel deep beneath surface through rock from a point* $A = (-1,0)$ *to a point* $B = (1,1)$ *along a curve* Γ*. Geological observations show that the rock mass density of the region can be modeled by*

$$f(x_1, x_2) = \exp(x_1 + x_2).$$

The costs and abbrasion of the tools are therefore proportional to the curve integral

$$\int_\Gamma f \, dl$$

Compute the integral along the straight line

$$\gamma : [0,1] \to \mathbb{R}^2, \quad t \mapsto (-1 + 2t, t)$$

Exercise 6 *A matrix* $A \in \mathbb{R}^{n\times n}$ *is called symmetric if* $A = A^t$ *and antisymmetric if* $A = -A^t$.

1. *Show that a matrix that is both symmetric and antisymmetric must be zero.*
2. *Show that every matrix* $B \in \mathbb{R}^{n\times n}$ *can be written as a sum* $A = A_1 + A_2$ *where* $A_1 \in \mathbb{R}^{n\times n}$ *is symmetric and* $A_2 \in \mathbb{R}^{n\times n}$ *is antisymmetric.*
3. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a scalar field with continuous partial derivatives up to second order. Show that its Hessian is a symmetric matrix.*
4. *The* trace *of a matrix is the sum of its diagonal elements. Find a scalar field in* $\mathbb{R}^2$ *whose Hessian is not zero but has zero trace.*
5. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable vector field. What is the antisymmetric part of its Jacobian?*
6. *Let* $G : \mathbb{R}^3 \to \mathbb{R}^3$ *be a differentiable vector field. What is the antisymmetric part of its Jacobian?*
7. *What do you notice in the answers to the last two questions?*