Analysis III - 202(c)

Winter Semester 2024

Session 2: September 19, 2024

Exercise 1 *Sketch the level sets of the functions*

$$f(x_1, x_2) = x_1 + x_2, \quad g(x_1, x_2) = x_1^2 x_2$$

for the values 0, 1, *and* -1. *Compute the gradients* ∇f *and* ∇g.

Exercise 2 *Compute the curl and the divergence of the following vector field:*

$$\vec{f}(x_1, x_2, x_3) = \left(\; x_1 x_2, \; x_3^2 x_1, \; e^{x_1 - x_2} \; \right)$$

Exercise 3 *Show that for every three-dimensional vector field* $\vec{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *we have*

$$\nabla \cdot \left(\nabla \times \vec{f} \right) = 0$$

Show that for every scalar field $g\colon \mathbb{R}^3 \to \mathbb{R}$ *we have*

$$\nabla \times (\nabla g) = 0$$

Consider the following vector field:

$$\vec{A}(x_1, x_2, x_3) := (\; \sin(x_2), \; \sin(x_3), \; \sin(x_1) \;)$$

Is this vector field a gradient of some scalar field?

Exercise 4 *Given scalar fields* $f, g : \mathbb{R}^3 \to \mathbb{R}$ *and vector fields* $\vec{A}, \vec{B} : \mathbb{R}^3 \to \mathbb{R}^3$, *show that*

$$\nabla \left(f \cdot g \right) = f \cdot \nabla g + g \cdot \nabla f,$$
$$\nabla \cdot \left(f \cdot \vec{A} \right) = (\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A}),$$
$$\nabla \cdot \left(\vec{A} \times \vec{B} \right) = \left(\nabla \times \vec{A} \right) \cdot \vec{B} - \vec{A} \cdot \left(\nabla \times \vec{B} \right),$$
$$\Delta(fg) = f \Delta g + 2 \nabla(f) \cdot \nabla(g) + g \Delta f.$$

Suppose that we also have two real numbers $\alpha, \beta \in \mathbb{R}$. *Show that*

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g,$$
$$\nabla \cdot (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \cdot \vec{A} + \beta \nabla \cdot \vec{B},$$
$$\nabla \times (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \times \vec{A} + \beta \nabla \times \vec{B}.$$

Exercise 5 *Consider the scalar field*

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \|x\|^{-1}$$

and compute its gradient and Laplacian. Show that Φ *is harmonic if* $n = 3$. *When* $n = 3$, *then this is the graviational potential, and its gradient is the gravitational field.*