11. 12. 2024

Poisson problem

We apply the superposition principle as follows:

Suppose we have a Poisson problem over $[a,b]$

$$-\Delta u(x) = f(x), \qquad a < x < b$$

$$u(a) = g_a, \qquad u(b) = g_b$$

We consider two auxiliary problems:

$$\left.\begin{aligned} -\Delta u^g(x) &= 0, \qquad a < x < b \\ u^g(a) = g_a, &\qquad u^g(b) = g_b \end{aligned}\right\} \mathrm{I}$$

$$\left.\begin{aligned} -\Delta u^f(x) &= f(x), \qquad a < x < b \\ u^f(a) = 0, &\qquad u^f(b) = 0 \end{aligned}\right\} \mathrm{II}$$

If we solve (I) and (II), then the sum $u := u^f + u^g$ solves the original Poisson problem.

(I) Consider the Poisson problem

$$-\Delta u^g = 0 \quad \text{over} \quad (a,b)$$

$$u^g(a) = g_a, \qquad u^g(b) = g_b$$

We know $-u''(x) = 0$ for all $a<x<b$. Hence $u''=0$ over (a,b), and so $u'(x)$ is constant over (a,b). Hence $u(x)$ is linear over (a,b). It has the form

$$u^g(x) = C_1 x + C_2$$

with some coefficients $C_1, C_2 \in \mathbb{R}$. We find C_1, C_2 using the boundary data:

$$g_a = C_1 a + C_2, \qquad g_b = C_1 b + C_2$$

We then solve a 2×2 linear system of equations.

Example: Let $[a,b] = [0,1]$. We solve

$$-\Delta u(x) = 0, \qquad 0 < x < 1$$

$$u(0) = 2, \qquad u(1) = 3$$

Then

$$u(x) = C_1 x + C_2$$

satisfies

$$2 = u(0) = C_1 \cdot 0 + C_2 \quad \Rightarrow \quad C_2 = 2$$

$$3 = u(1) = C_1 \cdot 1 + 2 \quad \Rightarrow \quad C_1 = 1$$

(II) We want to solve the Poisson problem over $[a,b]$:

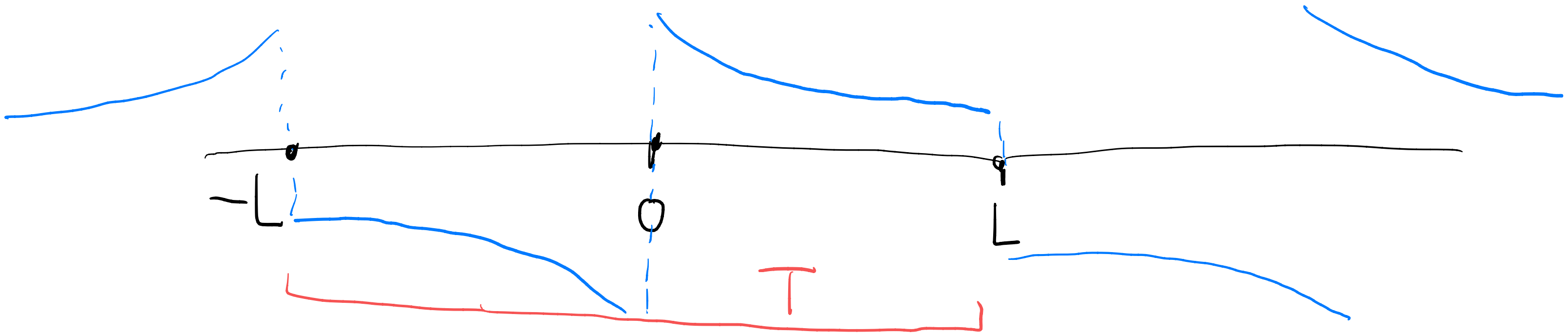
$$-\Delta u(x) = f(x) \qquad a < x < b$$

$$u(a) = 0, \qquad u(b) = 0$$

We use the Fourier series to solve this!

For simplicity, we assume $[a,b] = [0, L]$ where $L > 0$ is the length of the interval.

We extend f to an odd periodic function with period $T = 2L$



We study the Fourier series of this odd T-periodic extension of the source term f: the Fourier series is a sine Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{2\pi n}{T} x\right)$$

$$T = 2L$$

$$= \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{\pi n}{L} \cdot x\right)$$

For computations, we recall

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{\pi n}{L} x\right) dx \qquad \text{use that } f \text{ is } 2L\text{-periodic}$$

$$= \frac{2}{L} \int_0^{L} f(x) \sin\left(\frac{\pi n}{L} x\right) dx \qquad \text{use } f \text{ is odd}$$

Recall that

$$u(x) = \frac{L^2}{\pi^2 n^2} \sin\left(\frac{\pi n}{L} x \right)$$

satisfies

$$-u''(x) = \sin\left(\frac{\pi n}{L} x \right)$$

$$u(0) = 0, \qquad u(L) = 0$$

So we know how to solve the Poisson problem with zero Dirichlet BC if the source term is a single sine mode.

More generally, we can solve this problem when the source term is an infinite sum of sine modes (superposition principle)

We consider the function

$$u^f(x) = \sum_{n=1}^{\infty} b_n \frac{L^2}{\pi^2 n^2} \sin\left(\frac{\pi n}{L} x \right)$$

Since $u^f(x)$ is given by a Fourier sine series,

$$u^f(0) = 0, \qquad u^f(L) = 0$$

and take distribute the derivatives over each summand in the Fourier sine series:

$$-\Delta u^f(x) = \sum_{n=1}^{\infty} b_n \frac{L^2}{\pi^2 n^2} (-1) \Delta \left[\sin\left(\frac{\pi n}{L} x \right) \right]$$

$$= \sum_{n=1}^{\infty} b_n \cancel{\frac{L^2}{\pi^2 n^2}} \cancel{\frac{\pi^2 n^2}{L^2}} \sin\left(\frac{\pi n}{L} x \right) = f(x)$$

This holds for $x \in \mathbb{R}$ but we are only interested in $x \in (0, L)$.

So we have obtained the solutions of

$$\left.\begin{aligned} -\Delta u^g &= 0 \\ u^g(a) = g_a, &\quad u^g(b) = g_b \end{aligned}\right\} \mathrm{I}$$

$$\left.\begin{aligned} -\Delta u^f &= f \\ u^f(a) = 0, &\quad u^f(b) = 0 \end{aligned}\right\} \mathrm{II}$$

Their sum $u(x) = u^g(x) + u^f(x)$ solves the full problem

$$-\Delta u = f$$

$$u(a) = g_a, \qquad u(b) = g_b$$

Remarks:

- we obtain u^f as a Fourier series; depending on the Fourier coefficients, we may or may not express u^f by an explicit formula. Sometimes, the Fourier series of u^f corresponds to a nice formula, sometimes not.

- We have used an <u>odd</u> T-periodic extension because then u^f has a Fourier sine series, which automatically satisfies the homogeneous Dirichlet boundary conditions.

- If we had used an even T-periodic extension, then we would have a Fourier cosine series for f and u^f, and u^f would satisfy the homogeneous Neumann boundary conditions

$$u^{f\prime}(a) = 0, \qquad u^{f\prime}(b) = 0$$

- If we had used an L-periodic extension instead of a $2L$-periodic extension, then f would generally have a full Fourier series, and the same would be true for u^f.
In that situation, u^f would satisfy periodic boundary conditions

$$u^f(a) = u^f(b), \qquad u^{f\prime}(a) = u^{f\prime}(b)$$

- The Fourier series of f might not converge everywhere (Dirichlet theorem) and the same is true for u.
Moreover, the differential equation $-u'' = f$ might only hold in the sense of distributions.

- Practically, we might compute only a partial Fourier series $F_N f$, and then obtain a partial Fourier series of the solution. Then we solve the Poisson problem approximately.

Example: Suppose $[a,b] = [0,1]$. We solve

$$-\Delta u(x) = f(x) \qquad 0<x<1$$

$$u(0) = 2, \qquad u(1) = 3$$

where

$$f(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ 0 & \frac{1}{2} \le x < 1 \end{cases}$$

We remember that $u_g(x) = x+2$ satisfies

$$-u''(x) = 0, \qquad 0<x<1$$

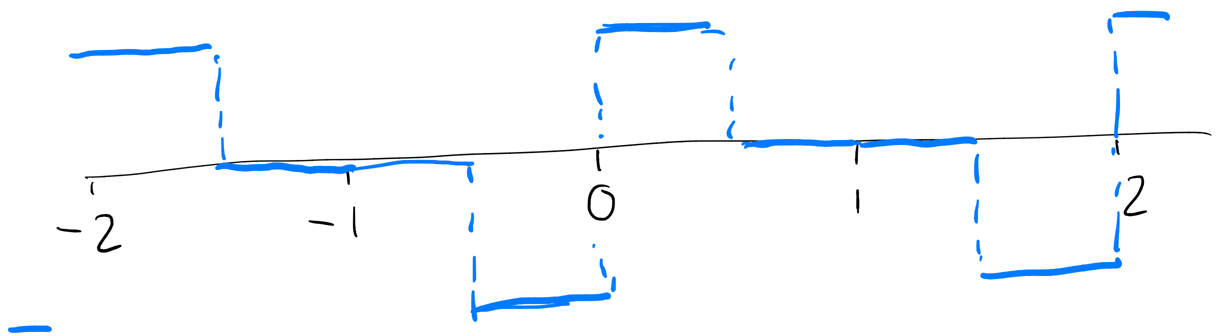
$$u(0) = 2, \qquad u(1) = 3$$

It remains to compute u_f satisfying

$$-u''(x) = f(x) \qquad 0<x<1$$

$$u(0) = 0, \qquad u(1) = 0$$

We extend f to an odd function $f: \mathbb{R} \to \mathbb{R}$ with period 2:



The formula reads:

$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ 0 & \frac{1}{2} \leq x < \frac{3}{2} \\ -1 & \frac{3}{2} \leq x < 2 \end{cases}$$

Since the extended f is an odd function,

it's Fourier series is a Fourier sine series

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{2\pi n}{2}x\right)dx$$

$$= \int_0^2 f(x)\sin(\pi n x)\,dx$$

$\leftarrow$ f is 2-periodic

$$= \int_{-1}^{+1} f(x)\sin(\pi n x)\,dx$$

$\leftarrow$ f is odd

$$= 2\int_0^1 f(x)\sin(\pi n x)\,dx$$

$$= 2\int_0^{1/2} \sin(\pi n x)\,dx$$

Hence

$$b_n = 2\left[\frac{-\cos(\pi n x)}{\pi n}\right]_{x=0}^{x=1/2} = \frac{2}{\pi n}\left(1 - \cos\left(\frac{\pi}{2}n\right)\right)$$

One possible way of rewriting these b_n :

For $n = 1, 2, 3, 4, 5, \ldots,$ the value $\cos(\frac{\pi}{2} n)$

cycles through $0, -1, 0, 1, 0, \ldots$

We then have $1 - \cos(\frac{\pi}{2} n)$ cycling through

$1, 2, 1, 0, 1, \ldots$

We can write

$$b_n = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even, } n/2 \text{ odd} \\ 0 & n \text{ even, } n/2 \text{ even} \end{cases}$$

Having found u^g and u^f, the full solution is:

$$u(x) = \underbrace{x + 2}_{u^g} + \underbrace{\sum_{n=1}^{\infty} b_n \frac{L^2}{\pi^2 n^2} \sin\left(\frac{\pi n}{L} x\right)}_{u^f}$$

$$= x + 2 + \sum_{n=1}^{\infty} b_n \frac{1}{\pi^2 n^2} \sin\left(\pi n x\right)$$

In summary:

- we have addressed the Poisson problem with Dirichlet boundary conditions over some finite interval
- we have used the superposition principle to split the original problem
  [possible because the Poisson problem is a linear differential equation]
- We have used a periodic odd extension to derive the solution formula in terms of a Fourier series. The computation can be done over the original interval.