12. 12. 2024

Poisson problem over $\mathbb{R}$

We have studied the Poisson problem over a finite interval. Instead, we now study the Poisson problem over the real line $\mathbb{R}$.

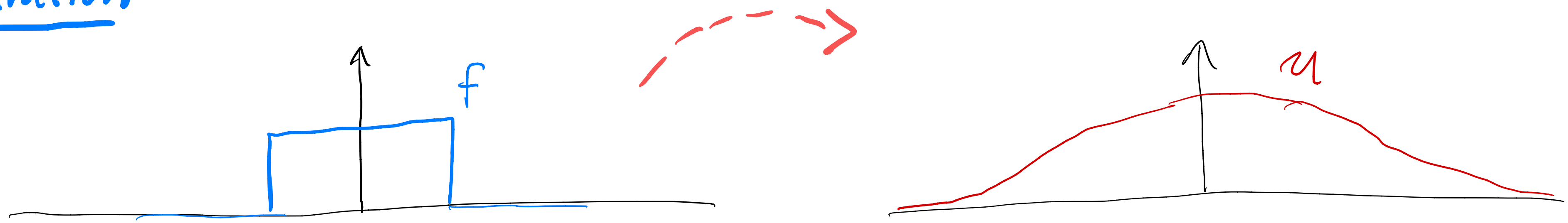
We study the Poisson problem with a zero-order term

$$-\Delta u(x) + k^2 u(x) = f(x)$$

for $-\infty < x < \infty$, where k is the so-called wave number. Again, f is called the source term.

No boundary conditions on u, since there is no boundary.
But we have a decay condition / "boundary condition towards infinity":
if f decays towards $\pm\infty$ sufficiently fast, then so does u.

Illustration



How to find a solution?

We find the solution using the Fourier transform

Conceptually: starting with the differential equation

$$-u''(x) + k^2 u(x) = f(x), \quad -\infty < x < +\infty$$

we apply the Fourier transform

$$-(i\alpha)^2 \mathcal{F}[u](\alpha) + k^2 \mathcal{F}[u](\alpha) = \mathcal{F}[f](\alpha)$$

using linearity of Fourier transform and its interaction with derivatives
We simplify

$$(\alpha^2 + k^2)\,\mathcal{F}[u](\alpha) = \mathcal{F}[f](\alpha)$$

We isolate $\mathcal{F}[u]$ and find

$$\mathcal{F}[u](\alpha) = (k^2 + \alpha^2)^{-1}\,\mathcal{F}[f](\alpha)$$

Since

$$\mathcal{F}[u](\alpha) = \frac{1}{k^2+\alpha^2}\mathcal{F}[f](\alpha)$$

we obtain by applying the inverse Fourier transform

$$u(x) = \mathcal{F}^{-1}\left[\frac{1}{k^2+\alpha^2}\mathcal{F}[f]\right](x)$$

We have found a solution formula for u:

- We need the FT of f
- We isolate the FT of u
- We need to compute the inverse FT

To compute with this, we need additional facts on the FT

II. Convolutions

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Their convolution

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

is another function $f * g : \mathbb{R} \to \mathbb{R}$.

Properties:

1) The convolution is linear in both functions

$$(a_1 f_1 + a_2 f_2) * g = a_1 (f_1 * g) + a_2 (f_2 * g)$$

$$f * (b_1 g_1 + b_2 g_2) = b_1 (f * g_1) + b_2 (f * g_2)$$

2) The convolution is commutative: $f * g = g * f$

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

$z = x - y$

$$= \int_{-\infty}^{+\infty} f(x-z)\, g(z)\, dz$$

$$= \int_{-\infty}^{+\infty} g(z)\, f(x-z)\, dz = (g * f)(x)$$

3) The convolution is associative:

$$(f * g) * h = f * (g * h)$$

$$[(f*g)*h](x) = \int_{-\infty}^{+\infty} (f*g)(y)\, h(x-y)\, dy$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(z)\, g(y-z)\, dz \right) h(x-y)\, dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z)\, g(y-z)\, h(x-y)\, dz\, dy$$

$$[f*(g*h)](x) = \int_{-\infty}^{+\infty} f(u)\, (g*h)(x-u)\, du$$

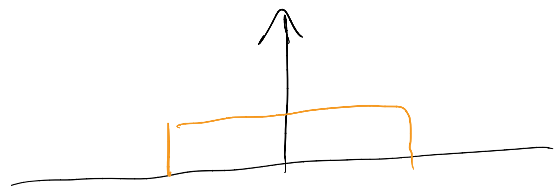
$$= \int_{-\infty}^{+\infty} f(u) \int_{-\infty}^{+\infty} g(v)\, h(x-u-v)\, dv\, du$$

$$= \int_{-\infty}^{+\infty} f(u) \int_{-\infty}^{+\infty} g(w-u)\, h(x-w)\, dw\, du$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u)\, g(w-u)\, h(x-w)\, dw\, du$$

Example: In many applications, the convolution is a weighted moving average. Suppose

$$g(x) = \begin{cases} 1/2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



Let $f: \mathbb{R} \to \mathbb{R}$ be some function. Then

$$(f * g)(x) = (g * f)(x)$$

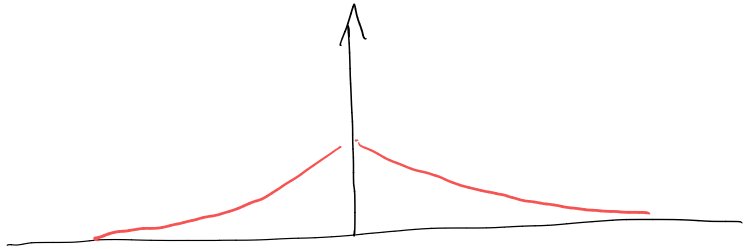
$$= \int_{-\infty}^{+\infty} g(y)\, f(x-y)\, dy$$

$$= \int_{-1}^{+1} \tfrac{1}{2} f(x-y)\, dy = \frac{1}{2} \int_{x-1}^{x+1} f(y)\, dy$$

Here, $(f * g)(x)$ is the average of f over $[x-1, x+1]$

Example Another weight is

$$g(x) = \frac{1}{2} e^{-|x|}$$



Note that g has integral 1.

$$(f * g)(x) = (g * f)(x) = \int_{-\infty}^{+\infty} \frac{e^{-|y|}}{2} f(x-y)\, dy$$

We interpret $f * g$ as another moving average:

$$(f * g)(x) = \int_{-\infty}^{-\infty} \frac{e^{-|x-z|}}{2} f(z)\, dz$$

4) Convolutions of Fourier transforms

If $h(x) = (f * g)(x)$, then

$$\hat{h}(\alpha) = \sqrt{2\pi}\, \hat{f}(\alpha) \cdot \hat{g}(\alpha)$$

If $h(x) = (f \cdot g)(x)$, then

$$\hat{h}(\alpha) = \sqrt{2\pi}\, \hat{f}(\alpha) * \hat{g}(\alpha)$$

In other words, the Fourier transform switches multiplication and convolution of functions. The inverse FT satisfies similar properties.

The factor $\sqrt{2\pi}$ depends on the particular convention of the FT.