Analysis III - Slides 02

Curves and curve integrals of scalar and vector fields

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Curve integrals

Le raison d'être of vector analysis

We remember integration as in high school:

$$\int_a^b f(x)\,dx \;=\; \text{integrate } f \text{ over interval } [a,b]$$

Now we generalize this:

$$\int_\gamma f\,dl \;=\; \text{integrate } f \text{ over curve } \gamma$$

We need to understand curves first...

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Curves

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Curves

A curve is a function

$$\gamma : [a, b] \longrightarrow \mathbb{R}^n, \quad t \longmapsto \gamma(t)$$

We may think of $\gamma(t)$ as the position at time t

The image of the curve γ is written Γ

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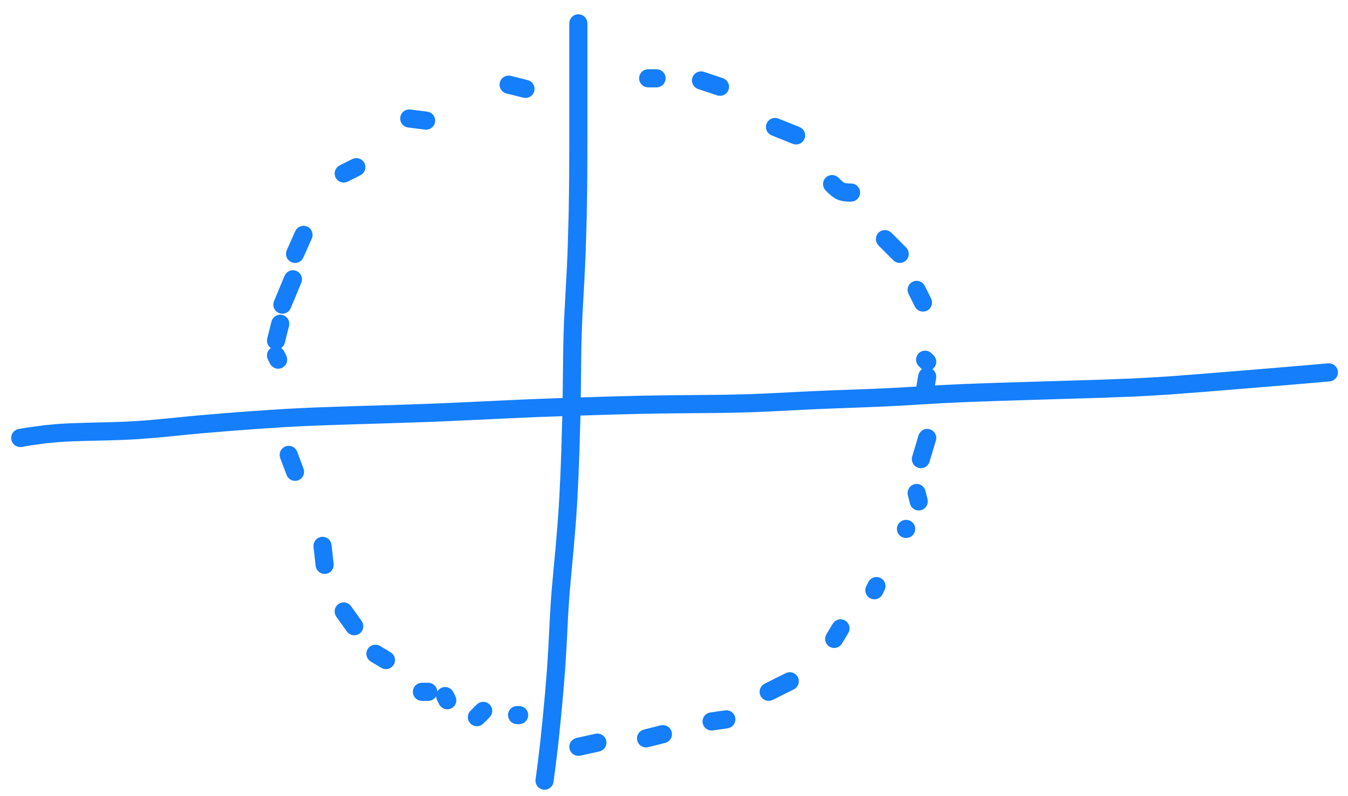
Example

$$\gamma : [0,1] \to \mathbb{R}^3, \quad t \mapsto (2+3t,\ 1+2t,\ 5-t)$$

Straight line from $(2,1,5)$ to $(5,3,4)$

$$\gamma : [0,2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos(t),\ \sin(t))$$



parameterization of unit circle

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Curves: Notions and definitions

We call a curve

$$\gamma : [a,b] \to \mathbb{R}^n, \quad t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$$

simple, if it does not self-intersect

formally, $\gamma : [a, b) \to \mathbb{R}^n$ is injective

closed, if $\gamma(a) = \gamma(b)$

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Curves: Notions and definitions

We call a curve

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n, \quad t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$$

differentiable, if $\gamma_1(t), \dots, \gamma_n(t)$ are differentiable

regular, if diff'able and $(\dot{\gamma}_1(t), \dots, \dot{\gamma}_n(t)) \neq \vec{0}$

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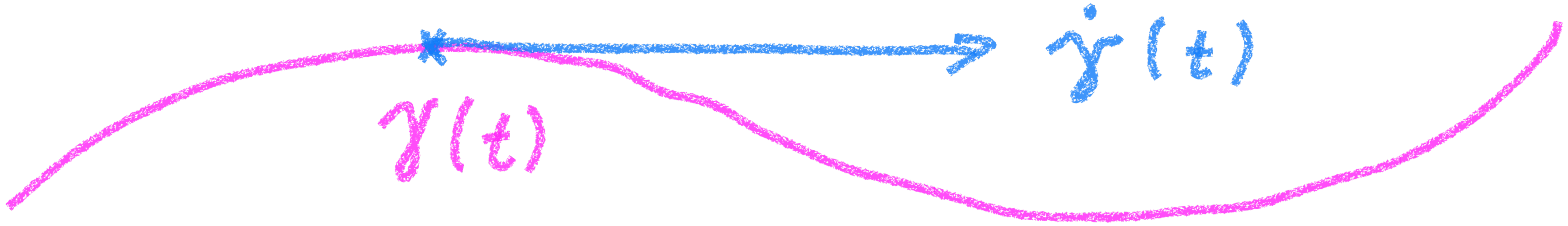
Curves: tangential vectors and speed

The tangent vector of a curve γ is

$$\dot{\gamma}(t) = \left(\dot{\gamma}_1(t), \ldots, \dot{\gamma}_n(t) \right)$$

and the speed is

$$|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}_1(t)^2 + \ldots + \dot{\gamma}_n(t)^2}$$



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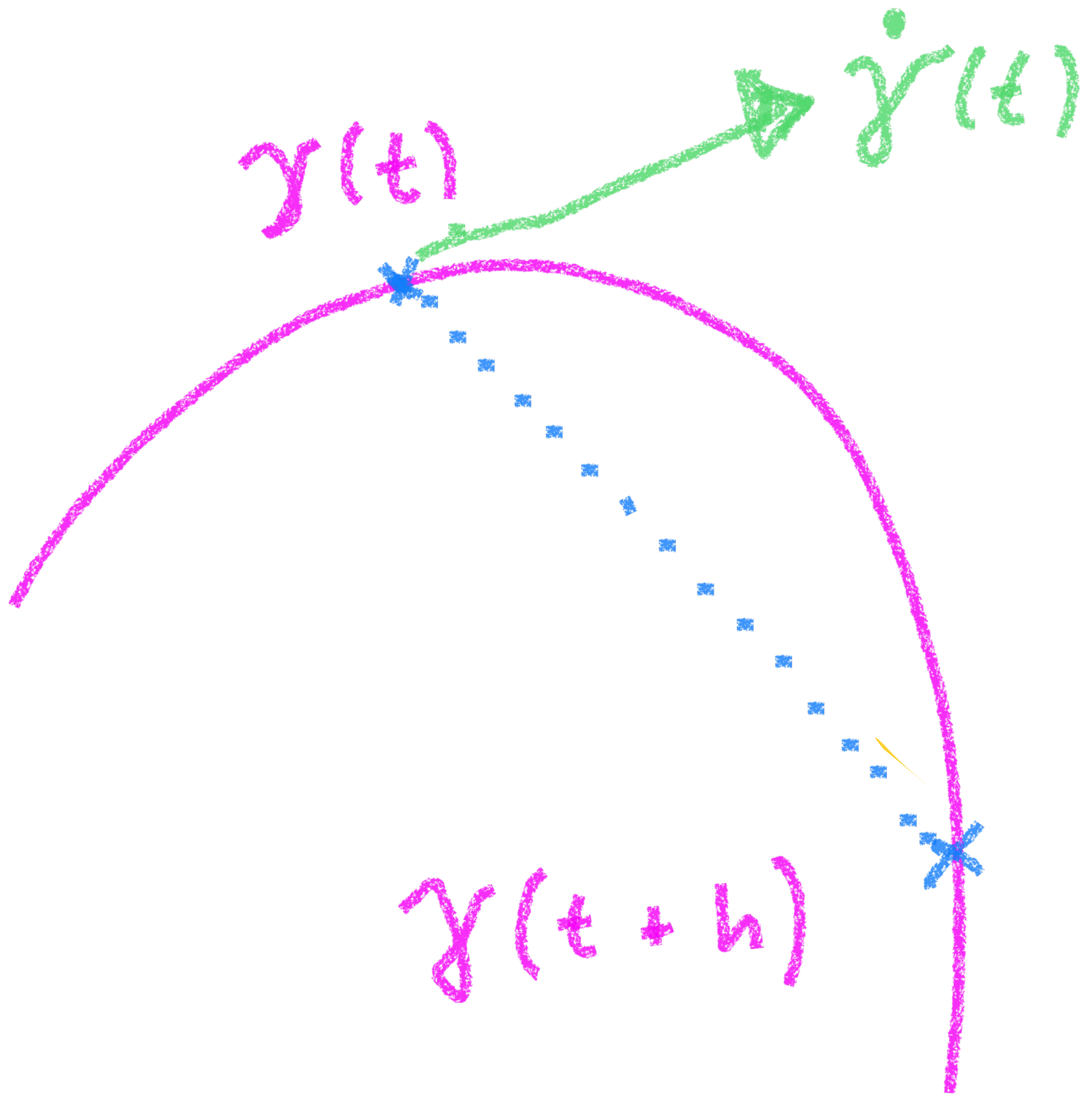
Curves: tangential vectors and speed



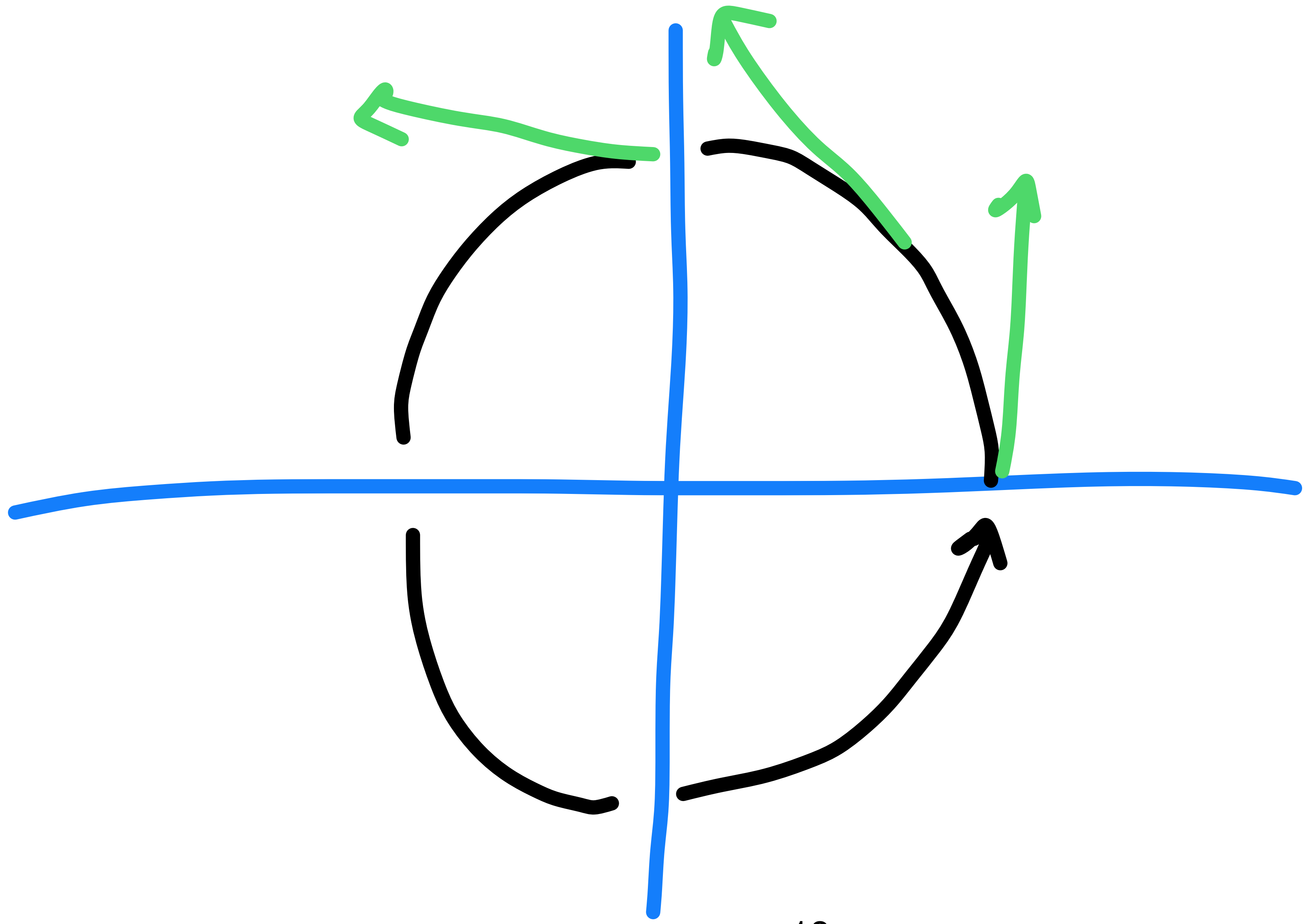
Similar to the definition of the derivative

$$\dot{\gamma}(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

Example

$$\gamma : [0, 2\pi) \to \mathbb{R}^2, \quad t \longmapsto (\cos(t), \sin(t))$$

$$\dot{\gamma}(t) = (-\sin(t), \cos(t))$$



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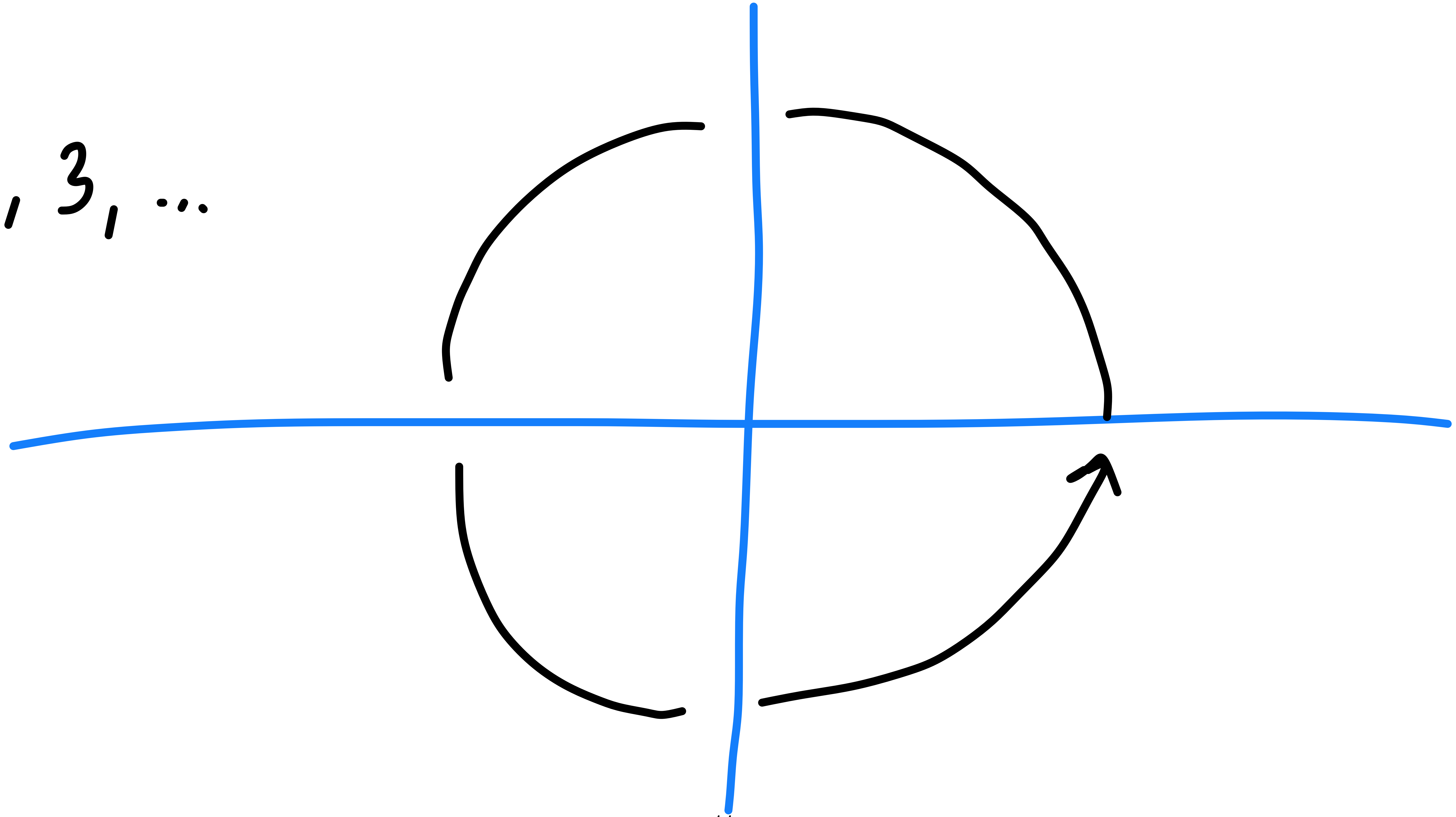
Example

$$\gamma : [0, k\pi) \to \mathbb{R}^2, \quad t \longmapsto (\cos(t), \sin(t))$$

$k = 1, 2, 3, \ldots$



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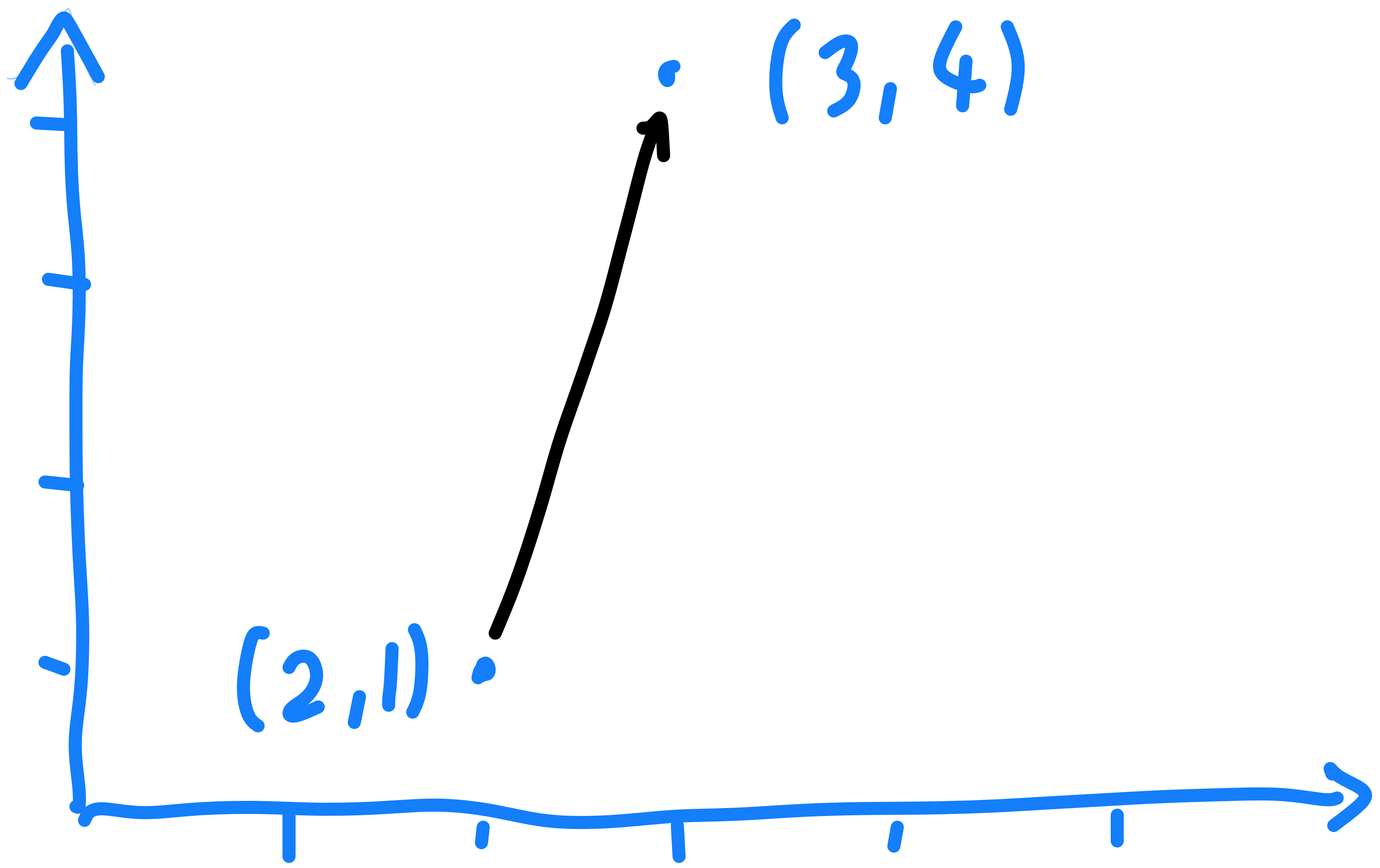
Example

$$\gamma : [0,1] \to \mathbb{R}^2, \qquad t \mapsto (2,1) + t(1,3)$$

$$\dot{\gamma}(t) = (1,3)$$



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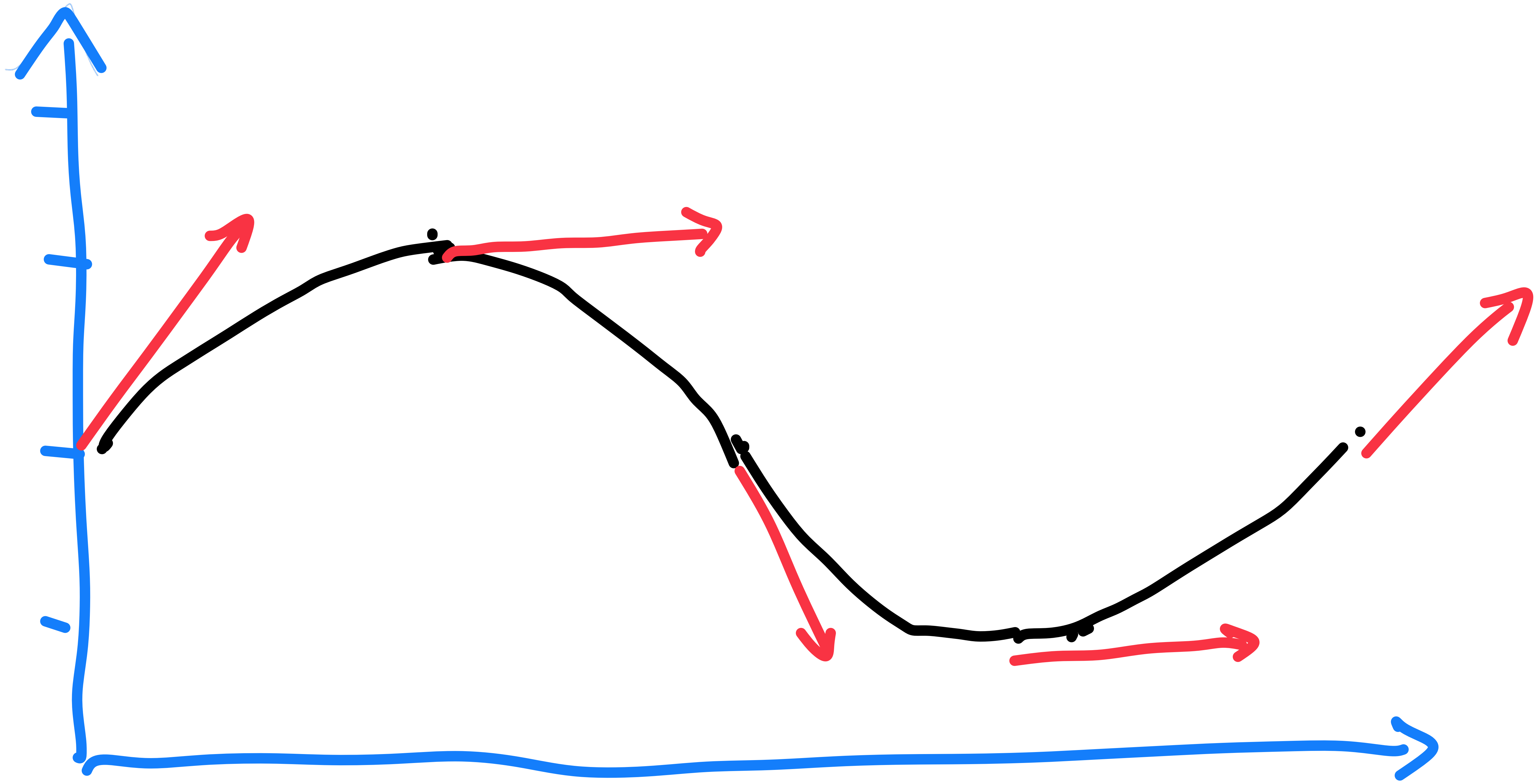
Example

$$\gamma : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t,\ 2 + \sin(t))$$

$$\dot{\gamma}(t) = (1,\ \cos(t))$$



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Curve integrals, explained

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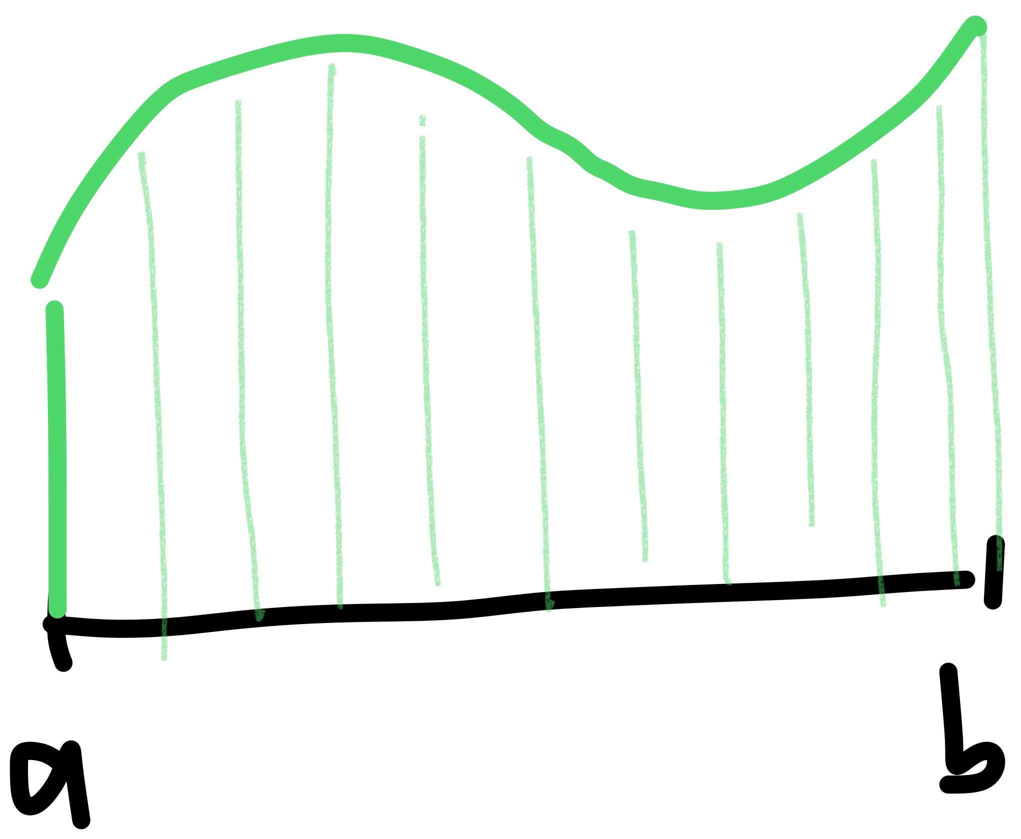
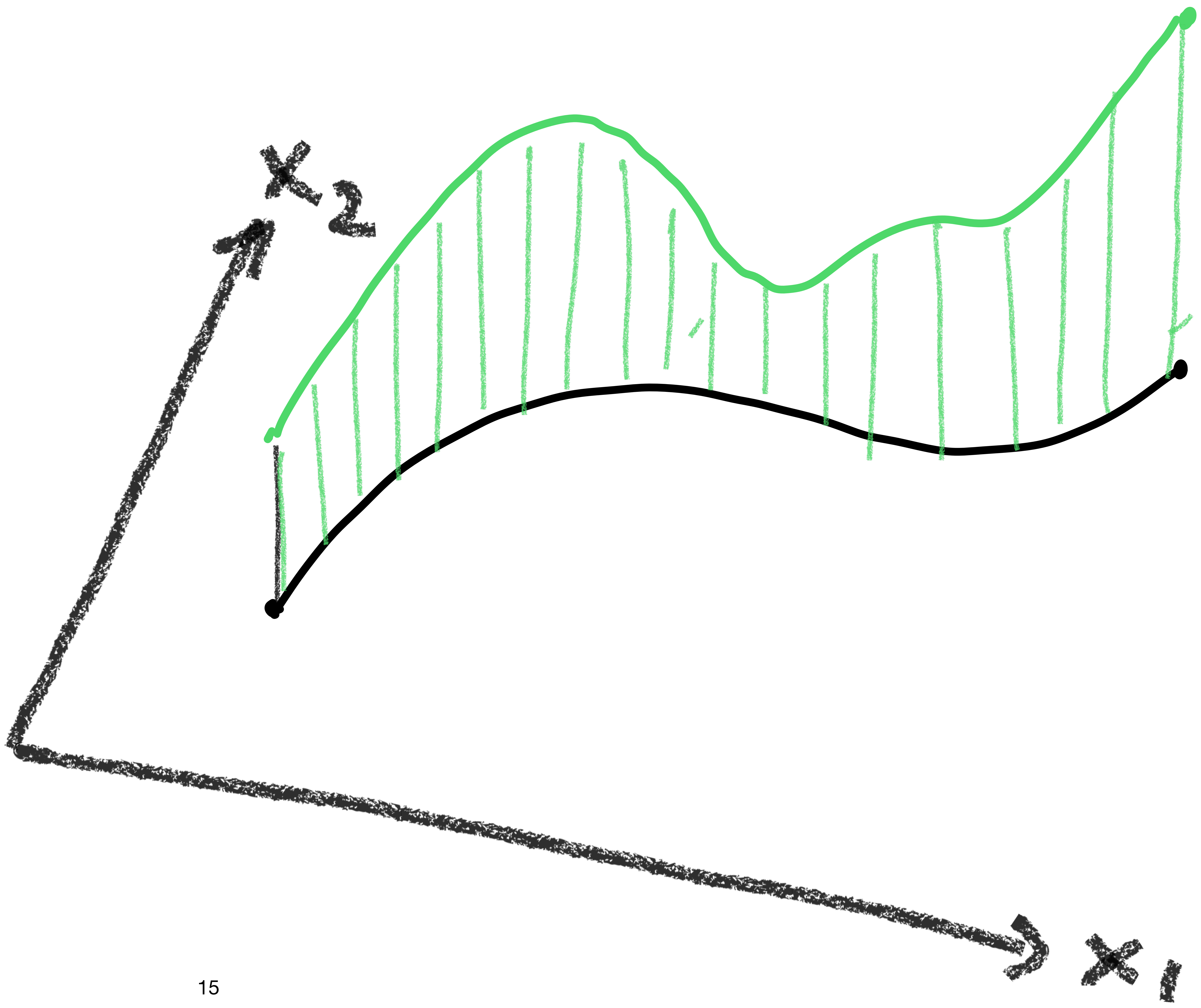
Curve integrals, explained

integral of f over [a,b]





Curve integrals, explained

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a curve

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field

The integral of f over Γ is

$$\int_\Gamma f \, d\ell := \int_a^b (f \circ \gamma)(t) \, |\dot{\gamma}(t)| \, dt$$

$$= \int_a^b (f \circ \gamma)(t) \sqrt{\dot{\gamma}_1(t)^2 + \ldots + \dot{\gamma}_n(t)^2} \, dt$$

Curve integrals, explained

$$\int_\Gamma f \, d\ell := \int_a^b (f \circ \gamma)(t) \; |\dot\gamma(t)| \; dt$$

The curve integral only depends on Γ, not on γ

- where γ is slow, $|\dot\gamma(t)|$ is small
- where γ is fast, $|\dot\gamma(t)|$ is large

$\Rightarrow$ fast sections emphasized, slow de-emphasized

$\Rightarrow$ no γ dependence

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Example

i) $f(x) = 1 \Rightarrow \int_\Gamma f \, dl = \int_a^b |\dot{\gamma}(t)| \, dt$

$= \text{length of } \gamma$

ii) $f(x) = 1$, $\gamma(t) = (2,3) + t(1,1)$

$0 \le t \le 1$

$\int_\Gamma f \, dl = \int_0^1 |(1,1)| \, dt$

$= \int_0^1 \sqrt{2} \, dt = \sqrt{2}$

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Curve integrals, explained

We need to distinguish between $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ and its image $\Gamma \subseteq \mathbb{R}^n$.

Many different parameterizations γ describe the same geometric curve Γ.

$$\gamma : [0,1] \longrightarrow \mathbb{R}^2, \quad t \longmapsto (1+t,\ 2+t)$$

$$\gamma : [0, \tfrac{\pi}{2}] \longrightarrow \mathbb{R}^2, \quad t \longmapsto (1+\sin(t),\ 2+\sin(t))$$

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Curve integrals, explained

Consider the function $f(x_1, x_2) = x_1 x_2$

$\gamma : [0,1] \to \mathbb{R}^2, \quad t \mapsto (1+t,\ 2+t)$

$$\int_\gamma f \, dl = \int_0^1 f(\gamma(t)) \cdot |\dot\gamma(t)| \, dt$$

$$= \int_0^1 (1+t)(2+t)\sqrt{2}\, dt = \sqrt{2}\int_0^1 2 + 3t + t^2 \, dt$$

$$= \sqrt{2}\left(2t + \frac{3}{2}t^2 + \frac{1}{3}t^3\right)\Big|_0^1 = \sqrt{2} \cdot 3\frac{5}{6}$$

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Curve integrals, explained

Consider the function $f(x_1, x_2) = x_1 x_2$

$\gamma : [0, \frac{\pi}{2}] \to \mathbb{R}^2, \quad t \mapsto (1 + \sin(t), \; 2 + \sin(t))$

$$\int_\gamma f \, dl = \int_0^{\pi/2} f(\gamma(t)) \cdot |\dot{\gamma}(t)| \, dt$$

$$= \int_0^{\frac{\pi}{2}} (1 + \sin(t))(2 + \sin(t)) \sqrt{2} \cos(t) \, dt$$

$u = \sin(t), \quad du = \cos(t) \, dt$

$$= \int_0^1 (1 + u)(2 + u) \sqrt{2} \, du = \dots$$

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Curve integrals, explained

If $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ is a simple regular curve then $\int_\Gamma f \, d\ell$ only depends on Γ.

As it should! After all, γ is just a parameterization and Γ is the "physical" object.

The last example is instructive: changing the parameterization corresponds to a change of variables.

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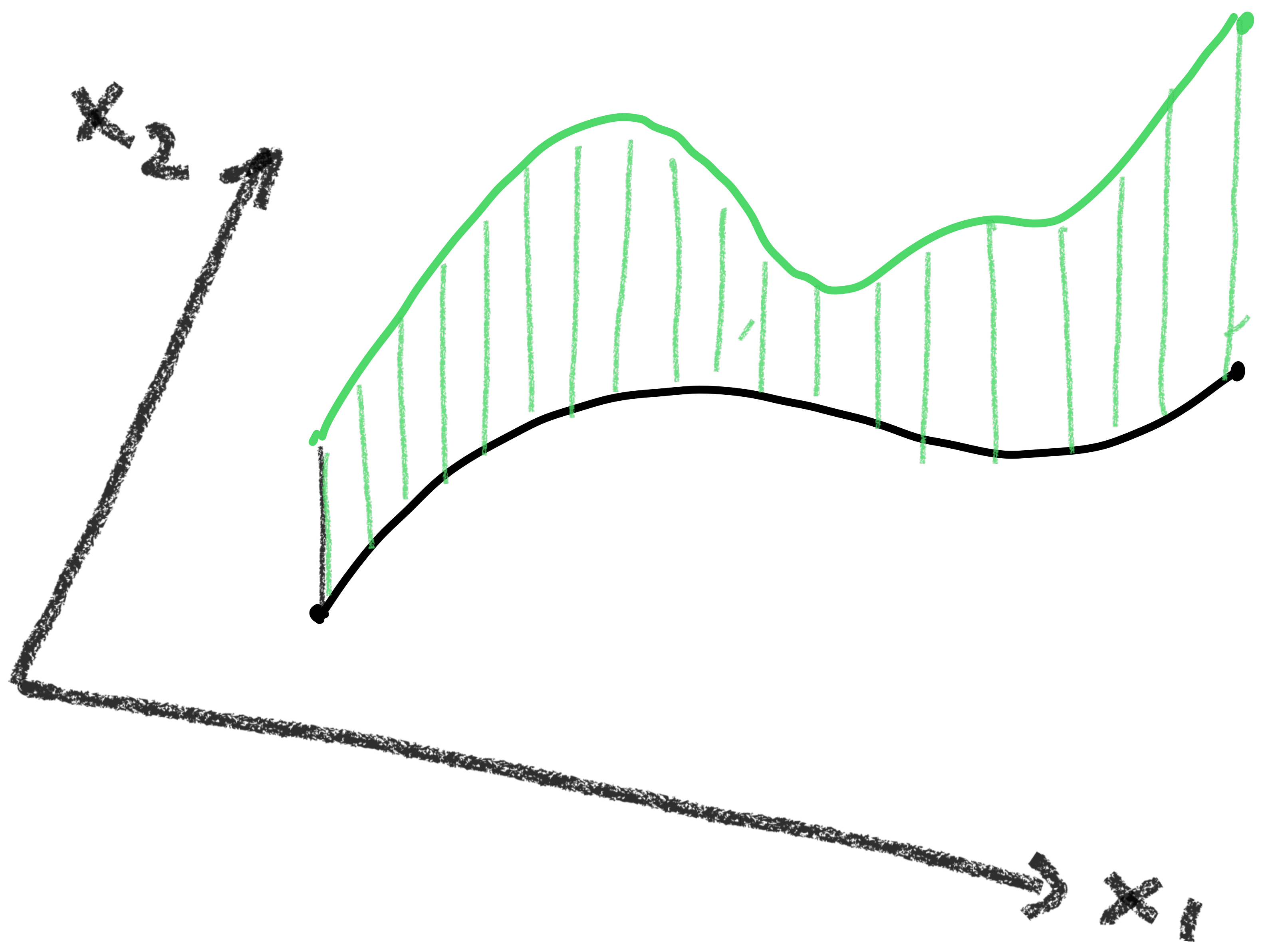
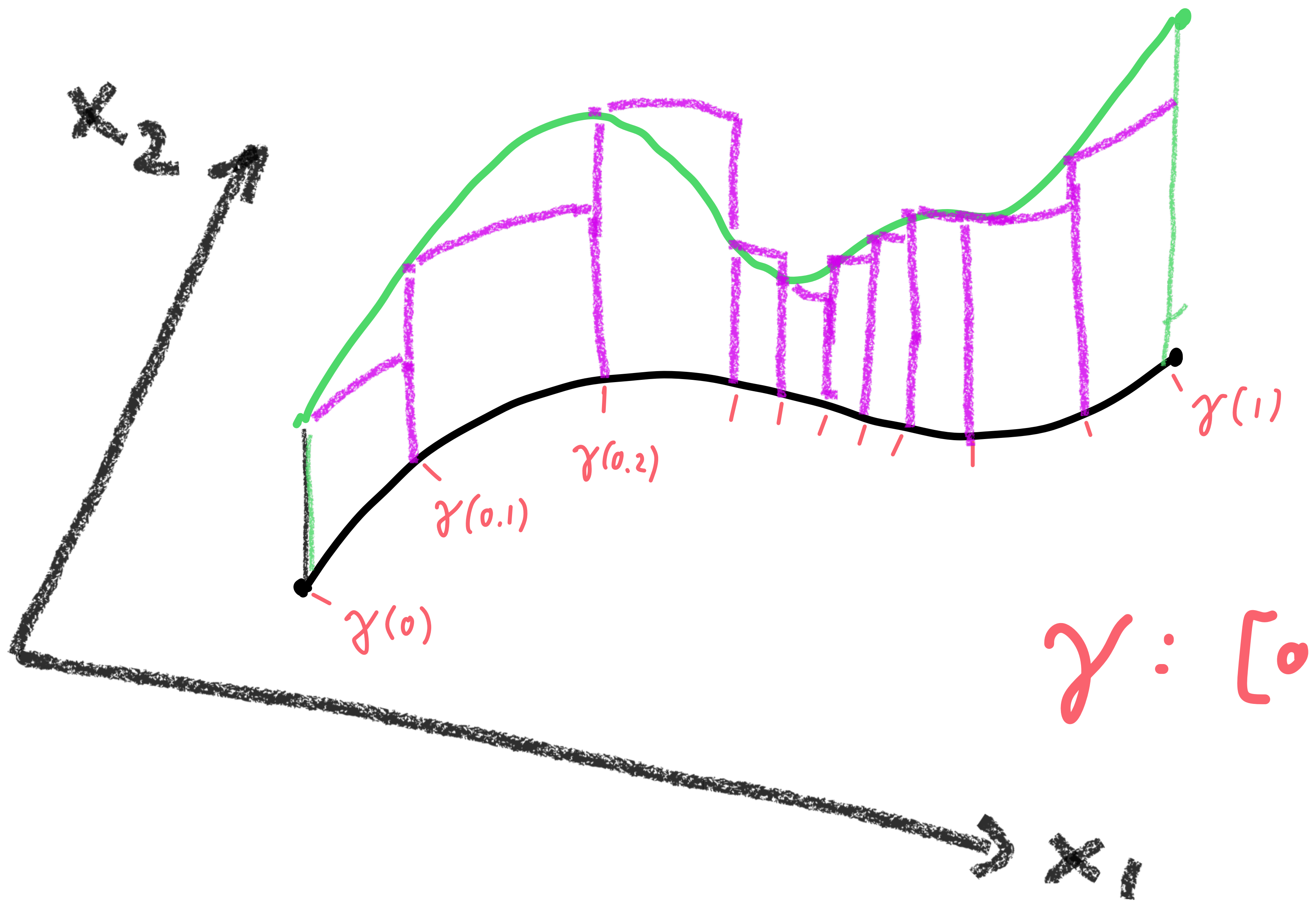
Curve integrals, explained

Some theory

Whatever the value of the curve integral, it should be the limit of the Riemann sum





$$\gamma : [0,1] \longrightarrow \mathbb{R}^2$$

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Curve integrals, explained

Some theory

We partition the interval $[a,b]$ at points $a = t_0, t_1, \ldots, t_N = b$

If we try to build the Riemann sum for the curve integral then a good start is

$$\int_\Gamma f\, d\ell \approx \sum_{i=1}^{N} f(\gamma(t_{i-1})) \cdot \text{length from } \gamma(t_{i-1}) \text{ to } \gamma(t_i)$$

$$\approx \sum_{i=1}^{N} f(\gamma(t_{i-1})) \cdot \|\gamma(t_i) - \gamma(t_{i-1})\|$$

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Curve integrals, explained

We show that this is another Riemann sum

$$\sum_{i=1}^{N} f(\gamma(t_{i-1})) \cdot \|\gamma(t_i) - \gamma(t_{i-1})\|$$

$$= \sum_{i=1}^{N} f(\gamma(t_{i-1})) \frac{\|\gamma(t_i) - \gamma(t_{i-1})\|}{t_i - t_{i-1}} (t_i - t_{i-1})$$

$$\dot{\gamma}(t_{i-1}) \approx \frac{\gamma(t_i) - \gamma(t_{i-1})}{t_i - t_{i-1}}$$

$$\approx \sum_{i=1}^{N} f(\gamma(t_{i-1})) \cdot \|\dot{\gamma}(t_{i-1})\| (t_i - t_{i-1})$$

$$\approx \int_a^b f(\gamma(t)) \|\dot{\gamma}(t)\| \, dt$$

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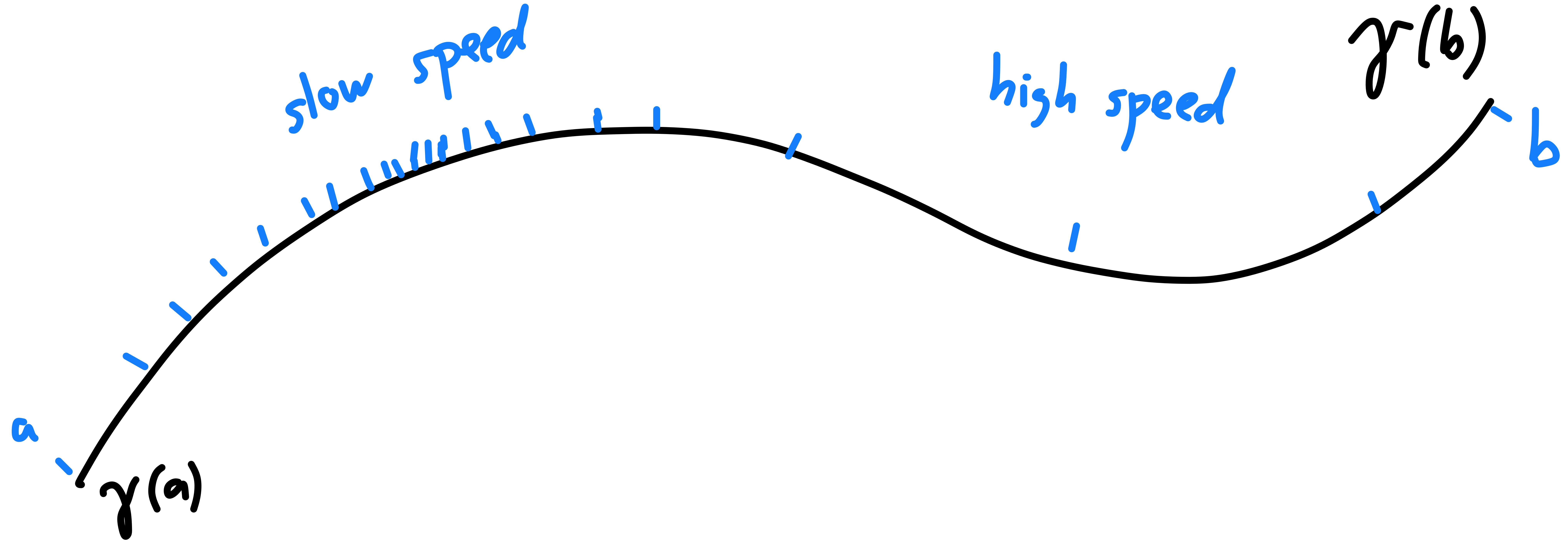
Curve integrals, explained



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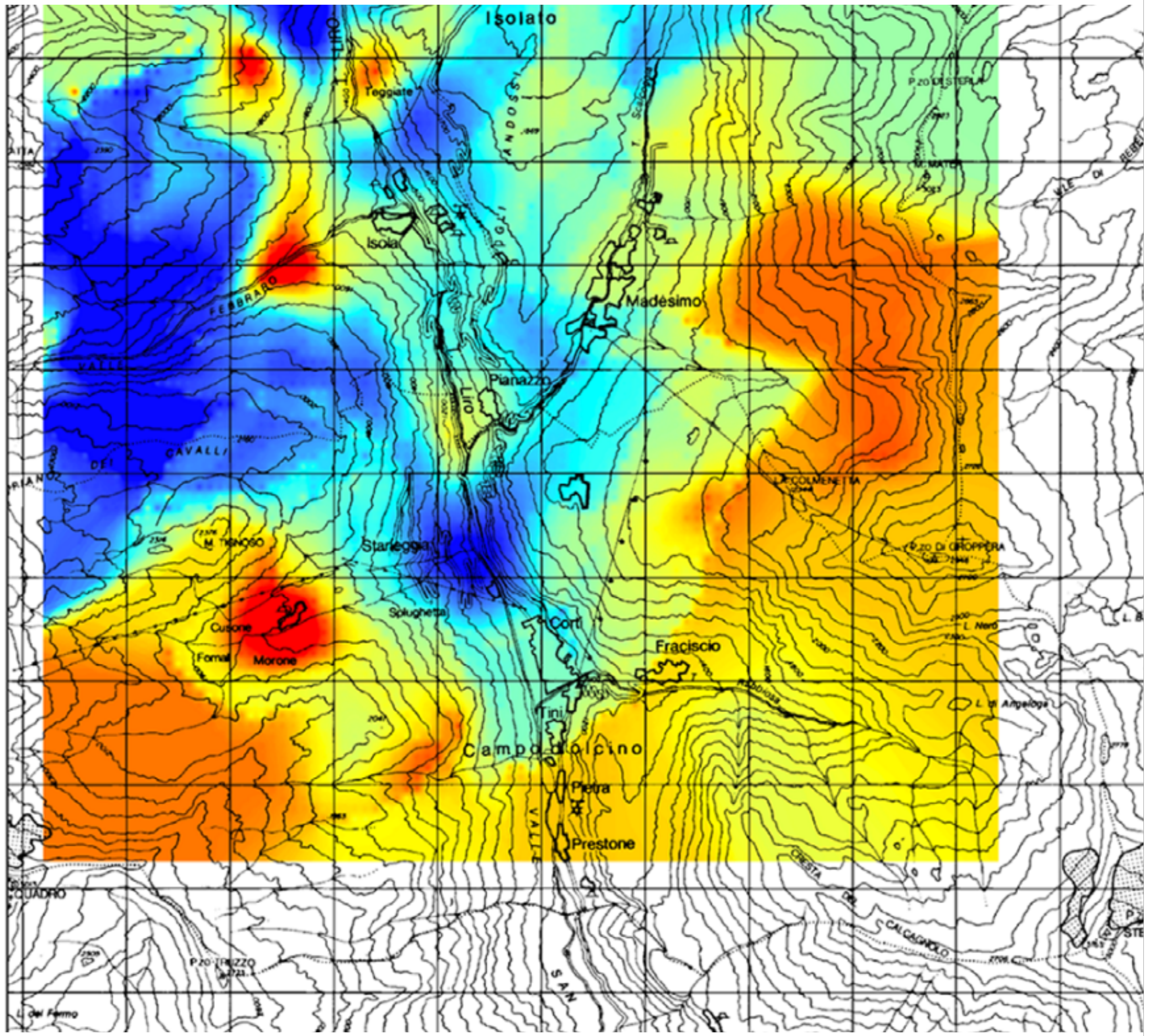
Curve integrals, explained

The function $f : \mathbb{R}^2 \to \mathbb{R}$ describes the rock mass density

Suppose that $\gamma : [0,1] \to \mathbb{R}^2$ describes the path of a tunnel deep beneath the surface

The effort in excavating a tunnel along Γ equals

$$\int_\Gamma f \, dl$$



Rock mass density in San Giacomo Valley, north of Milano

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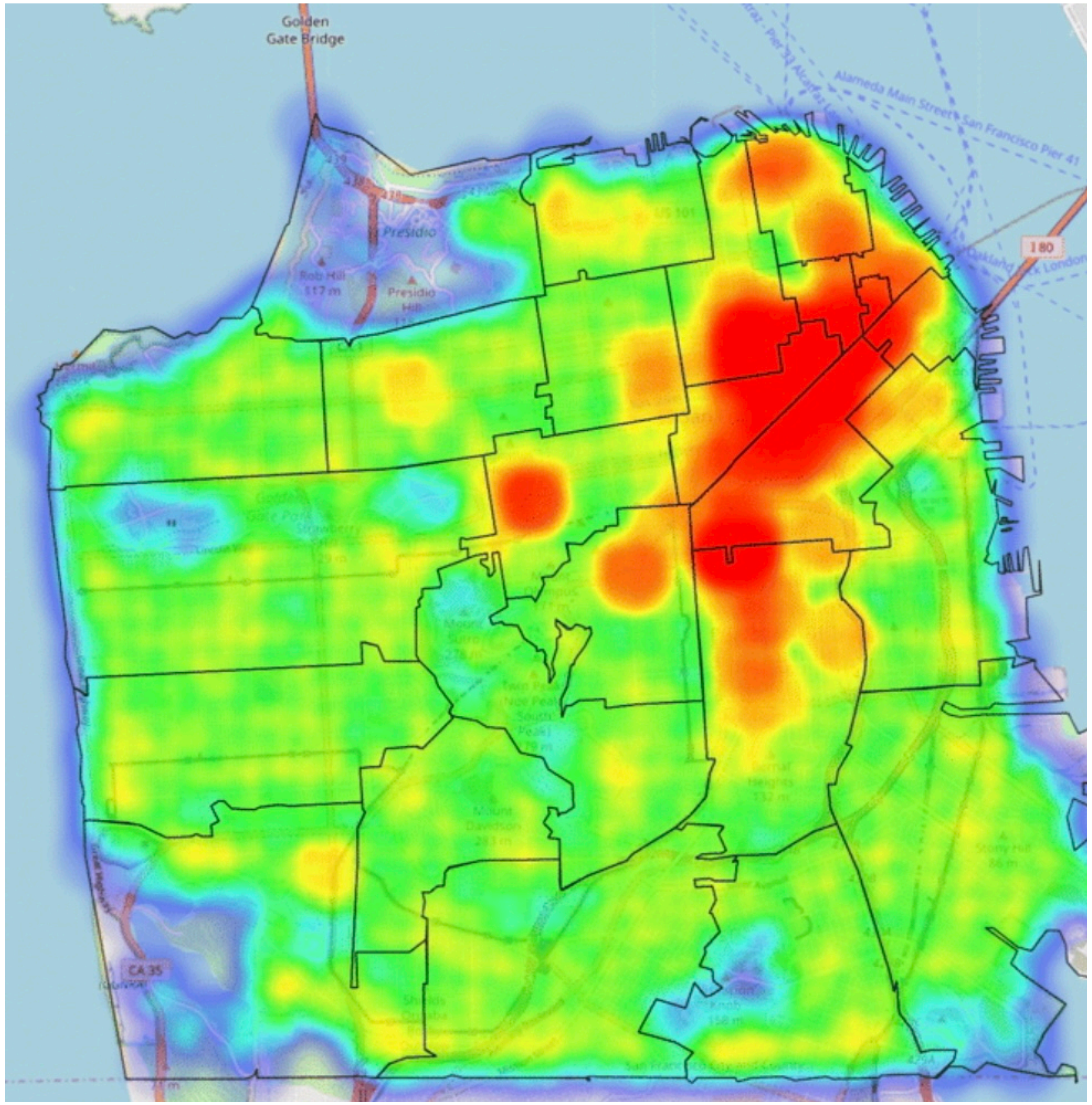
Curve integrals, explained

The function $\rho : \mathbb{R}^2 \to \mathbb{R}$ describes the crime density

Before you walk there along a curve Γ, you may consider the integral

$$\int_\Gamma \rho \, dl$$



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Curve integrals of vector fields

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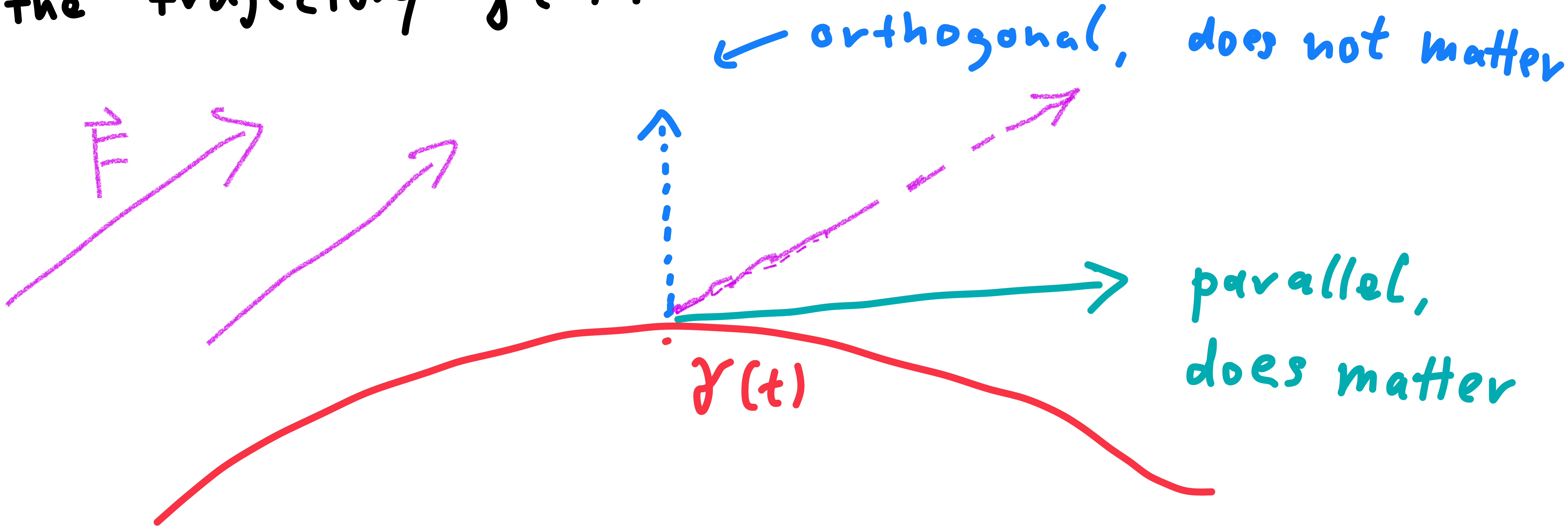
Curve integrals of vector fields

Suppose a train moves along a path $\gamma : [a,b] \to \mathbb{R}^3$ through wind that flows along a vector field $\vec{F}$

What accelaration / deaccelaration is needed to keep the trajectory $\gamma(t)$?



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Curve integrals of vector fields

Given a curve $\gamma : [a,b] \to \mathbb{R}^n$ and a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, we define

$$\int_\Gamma \vec{F}\, d\ell := \int_a^b \vec{F}(\gamma(t)) \cdot \dot{\gamma}(t)\, dt$$

scalar / dot product

$$= \int_a^b F_1(\gamma(t))\, \dot{\gamma}_1(t) + \dots + F_n(\gamma(t))\, \dot{\gamma}_n(t)\, dt$$

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Example

Suppose $\vec{F} = \nabla f$. Then

$$\int_\Gamma \nabla f \, d\ell = \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t)\, dt$$

$$= \int_a^b (f \circ \gamma)'(t)\, dt$$

$$= f(\gamma(b)) - f(\gamma(a))$$

We call f the "potential function" of the vector field F

F is a "conservative" vector field

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Example (Kinetic energy)

Let $\vec{F}$ be a force field accelarating a particle.
The path γ satisfies the differential equation

$$\ddot{\gamma}(t) = \vec{F}(\gamma(t))$$

$$\Rightarrow \quad \ddot{\gamma}(t)\cdot\dot{\gamma}(t) = \vec{F}(\gamma(t))\cdot\dot{\gamma}(t)$$

$$\Rightarrow \quad \int_a^b \ddot{\gamma}(t)\cdot\dot{\gamma}(t)\,dt = \int_a^b \vec{F}(\gamma(t))\,\dot{\gamma}(t)\,dt$$

$$\int_a^b \tfrac{1}{2}\left(\gamma'(t)\cdot\gamma'(t)\right)'dt = \tfrac{1}{2}\dot{\gamma}(b)^2 - \tfrac{1}{2}\dot{\gamma}(a)^2 = \int_\Gamma \vec{F}\,dl$$

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Example

$$\vec{F}(x,y) = (-y, x)$$

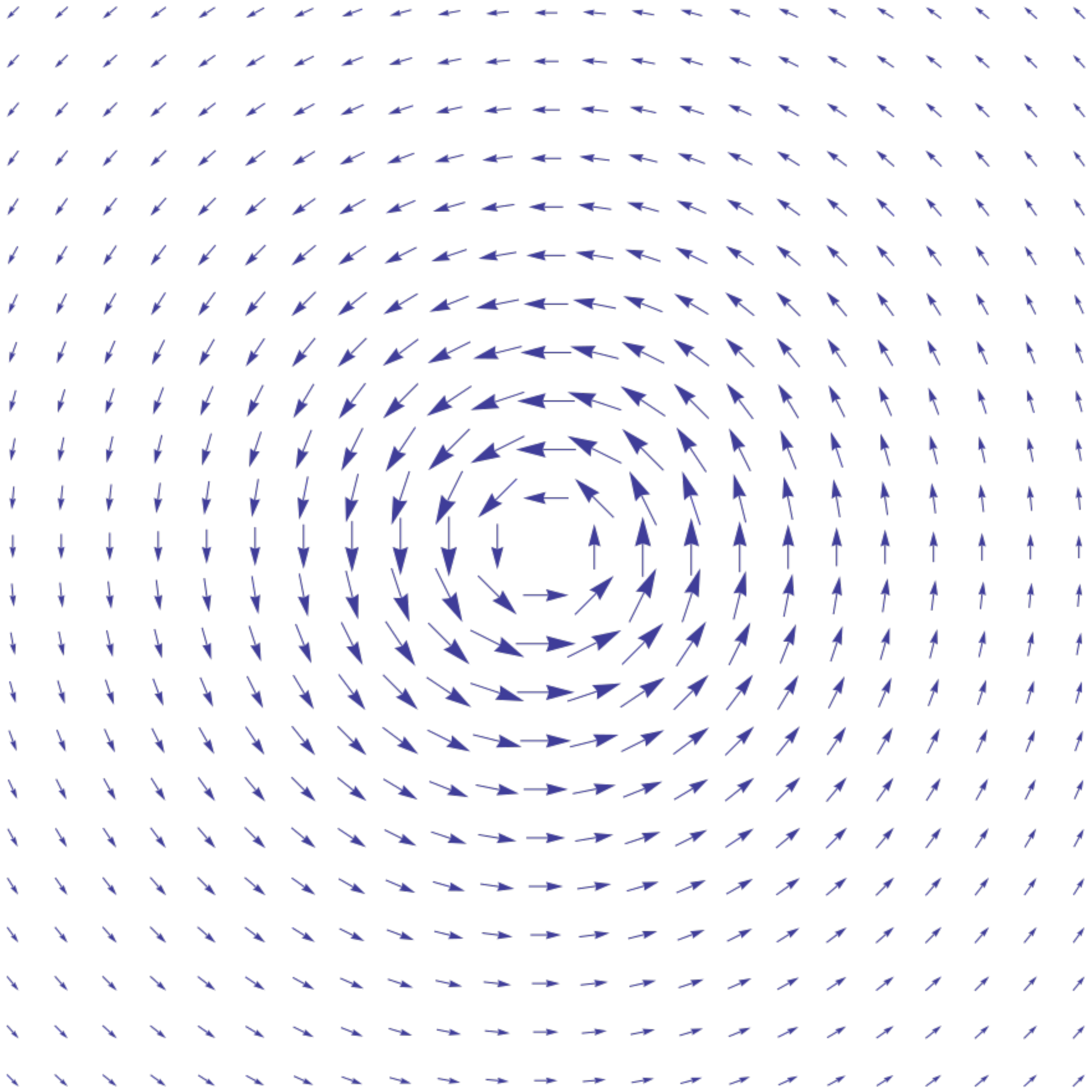
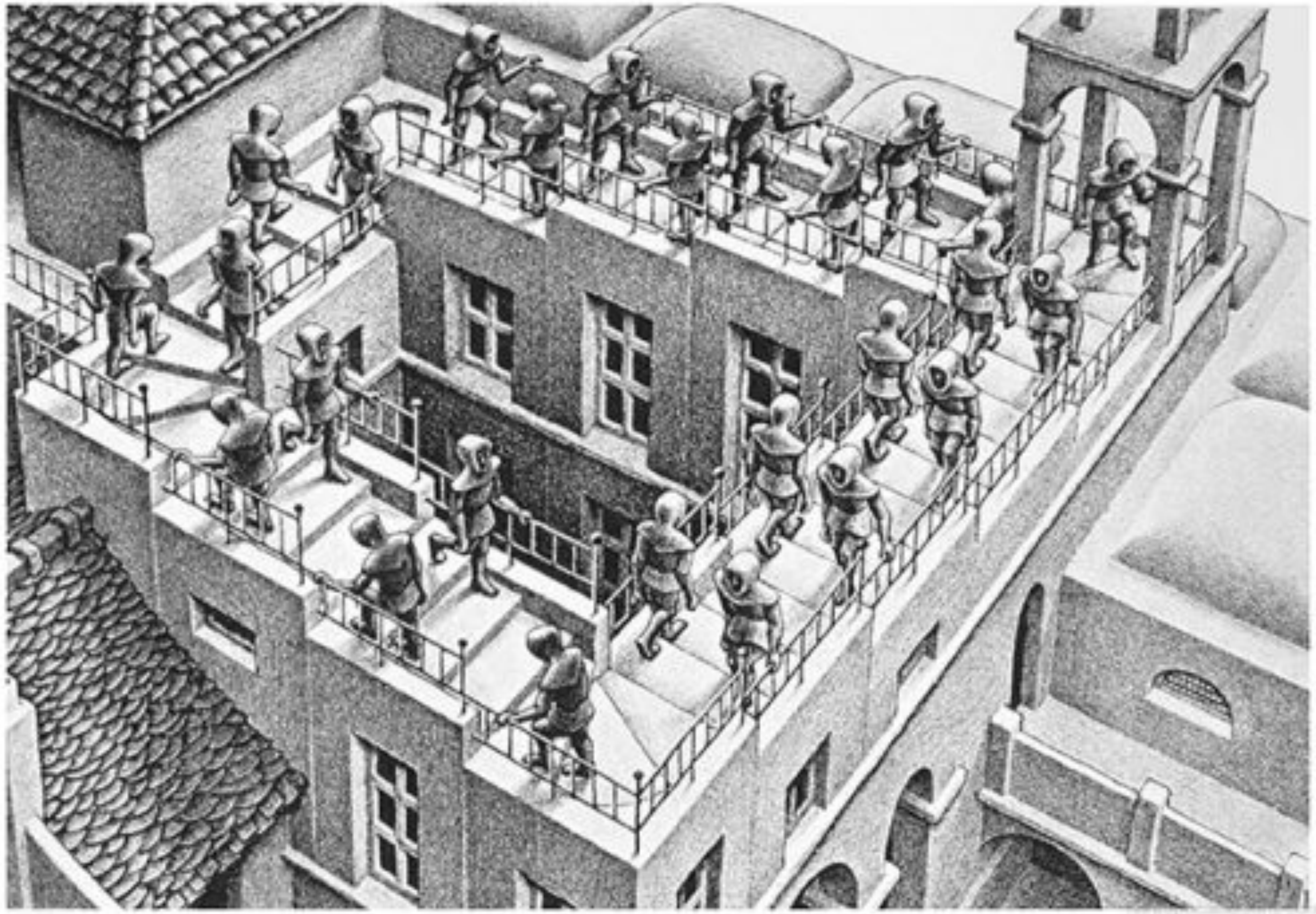
$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$$

$$\int_\Gamma \vec{F}\, dl = \int_0^{2\pi} \vec{F}(\gamma(t))\, \dot{\gamma}(t)\, dt$$

$$= \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$$

$\Rightarrow$ F is not a gradient

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Summary and outlook

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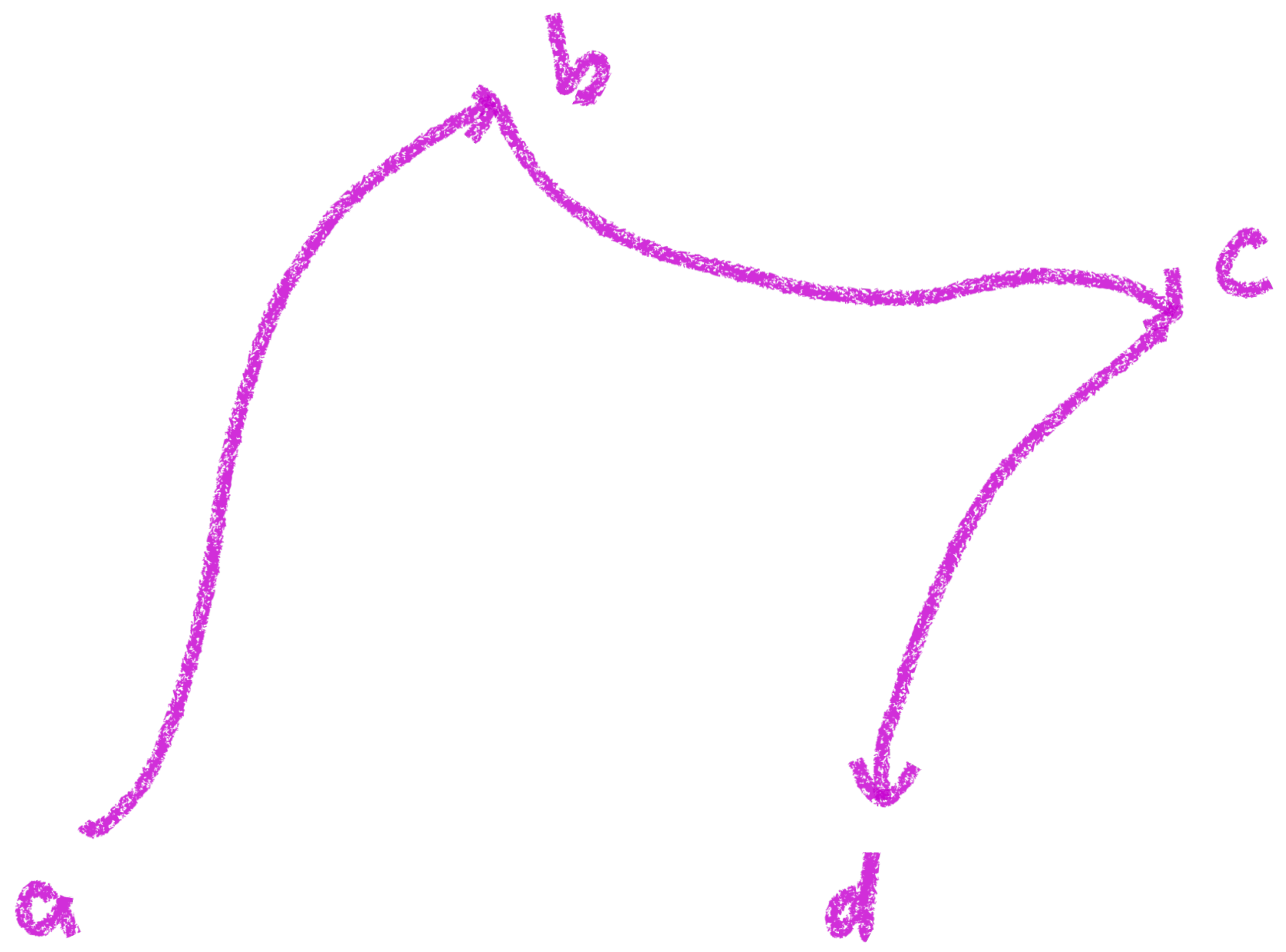
Instead of simple regular differentiable curves, we will often use curves that simple and piecewise regular differentiable.



A simple curve $\gamma : [a,d] \to \mathbb{R}^2$ regular and differentiable over $[a,b)$, $[b,c]$, $[c,d]$

$$\int_\Gamma f \, d\ell = \int_a^b f\circ\gamma \, |\dot{\gamma}| \, dt + \int_b^c f\circ\gamma \, |\dot{\gamma}| \, dt + \int_c^d f\circ\gamma \, |\dot{\gamma}| \, dt$$

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Curves, basic definitions, tangents

curve integrals of functions

curve integrals of vector fields

Next: conservative vector fields

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