EPFL - Fall 2024	Leonid Monin
Analysis III - 203 (a)	**Exercises**
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Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Aide-memoire: To verify the divergence theorem in $\mathbb{R}^3$, proceed as follows :

1. *Sketch the domain Ω, then calculate* $\operatorname{div} F(x,y,z)$.
2. *Parameterize the domain Ω. Use this parameterization to express*
$$\iiint_\Omega \operatorname{div} F(x,y,z)\, dxdydz$$
as a triple integral where the bounds and the function to be integrated are explicitly indicated.
3. *Write $\partial\Omega$ as a union of regular surfaces; for each, give a parameterization and a field of exterior normals. Add this to your sketch.*
4. *Express*
$$\iint_{\partial\Omega} F \cdot \nu\, ds$$
as a sum of double integrals where the bounds and the functions to be integrated are explicitly indicated.
5. *Check the conclusion of the divergence theorem for Ω and F.*

Exercise 1 (Ex 6.4 page 66)**.**
Verify the divergence theorem for $F(x,y,z) = (x^2, y^2, z^2)$ and
$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : b^2\left(x^2+y^2\right) < a^2z^2 \text{ and } 0 < z < b\right\}$.

Exercise 2 (Ex 6.3 page 66)**.**
Verify the divergence theorem for $F(x,y,z) = (xy, yz, xz)$ and
$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : 0 < z < 1-x-y,\ 0 < y < 1-x,\ 0 < x < 1\right\}$.

Exercise 3 (Ex 6.6 page 66)**.**
Verify the divergence theorem for $F(x,y,z) = (x,y,z)$ and
$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 4 \text{ et } x^2+y^2 < 3z\right\}$.

Exercise 4 (Ex 6.9 page 67).
Verify the divergence theorem for $F(x,y,z) = \left(2, 0, xy^2 + z^2\right)$ and
$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : x > 0,\ 0 < z < 2 \text{ et } 4\left(x^2 + y^2\right) < (z-4)^2\right\}$.

Exercise 5 (Ex 6.11 page 67).
Let $\Omega \subset \mathbb{R}^3$ a regular domain and ν a field of unit normals exterior to Ω. Let the vector fields F, G_1, G_2 and G_3 defined by

$$F(x,y,z) = (x,y,z), \qquad G_1(x,y,z) = (x,0,0),$$
$$G_2(x,y,z) = (0,y,0), \qquad G_3(x,y,z) = (0,0,z).$$

Show that:

1. $\text{Volume}(\Omega) = \dfrac{1}{3} \displaystyle\iint_{\partial\Omega} (F \cdot \nu)\, ds$
2. $\text{Volume}(\Omega) = \displaystyle\iint_{\partial\Omega} (G_i \cdot \nu)\, ds$ for $i = 1, 2, 3$.