

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1 (Ex 3.1 page 27).

Let $F_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector fields defined as :

$$F_1(x, y) = (y, xy - x), \quad F_2(x, y) = (3x^2y + 2x, x^3), \quad F_3(x, y) = (3x^2y, x^2).$$

Does the vector field F_i derive from a potential on $\mathbb{R}^2$?

If yes, find a potential function from which F_i is deriving. If not find a closed path Γ such that: $\int_{\Gamma} F_i \cdot dl \neq 0$.

Exercise 2 (Ex 3.3 page 28).

$$\text{Let } F(x, y, z) = \left(2xy + \frac{z}{1+x^2}, x^2 + 2yz, y^2 + \arctan x \right).$$

Does the vector field F derive from a potential on $\mathbb{R}^3$? If yes, find this potential.

Exercise 3 (Ex 3.8 page 29).

Let:

$$F(x, y) = \left(\frac{-x}{(x^2 + y^2)^2}, \frac{-y}{(x^2 + y^2)^2} \right),$$

and

$$G(x, y) = \left(\frac{y^3}{(x^2 + y^2)^2}, \frac{-xy^2}{(x^2 + y^2)^2} \right),$$

defined on $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Do they derive from a potential on Ω ? (If yes find a potential function, if no justify your answer.)

Exercise 4 (Ex 3.6 page 28).

Let us define the differential equation :

$$F_2(t, u(t)) u'(t) + F_1(t, u(t)) = 0 \quad \text{for } t \in \mathbb{R}.$$

Let $F(x, y) = (F_1(x, y), F_2(x, y))$ be a vector field which derives from a potential f on $\mathbb{R}^2$.

1. Show that a solution $u(t)$, in implicit form, of this differential equation is given by :

$$f(t, u(t)) = \text{constant} \quad \text{for all } t \in \mathbb{R}.$$

Indication: Calculate $\frac{d}{dt} f(t, u(t))$.

2. Deduce a solution of:

$$u^2(t)u'(t) + \sin t = 0, \text{ for } t \in \mathbb{R} \text{ with the initial condition } u(0) = 3.$$

Exercise 5 (Ex 3.2 page 27).

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, defined by $F(u, v) = (f(u, v), g(u, v))$.

Let:

$$\varphi(x, y) = \int_0^1 [xf(tx, ty) + yg(tx, ty)] dt.$$

1. Show that if $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial v}$, then $F(x, y) = \text{grad } \varphi(x, y)$.
2. Deduce a potential for $F(x, y) = (2xy, x^2 + y)$.
3. Generalize the result shown in question 1 to $\mathbb{R}^n$, *i.e.* in the case that:

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n; \quad u \mapsto F(u) = (F_1(u), \dots, F_n(u)).$$