Teacher : Teachers of Analysis III
Analysis III - Mock exam - Student
December 2021
Duration : 120 minutes

1

Student One

SCIPER : **111111**

Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.

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- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  $+2$ points if your answer is correct,
  0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

For your examination, preferably print documents compiled from auto-multiple-choice.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 All the non-zero real Fourier coefficients (sine-cosine form) of the function f defined by:

$$f(x) = 5 + \frac{1}{2}(5 - i\pi)e^{3ix} + \frac{1}{2}(5 + i\pi)e^{-3ix} - \frac{3}{2}e^{7ix} - \frac{3}{2}e^{-7ix}$$

are

- ☐ a_0, a_3, a_7
- ☐ a_3, a_7, b_3
- ☐ a_0, a_3, b_3, b_7
- ☐ $a_0,\ a_3,\ a_7,\ b_7$

Question 2 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\displaystyle\int_\Gamma f\,\mathrm{d}l$ is equal to:

- ☐ 0
- ☐ $2\pi R$
- ☐ $2\pi R^2 + \pi R + 1$
- ☐ $2\pi R^3 + \pi R^2 + R$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto (\cancel{x, y}) \quad (y, x),$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of A by ∂A, and the outer unit normal of ∂A by $\nu : \partial A \to \mathbb{R}^2$. The integral $\displaystyle\int_{\partial A} F \cdot \nu\, dl$ is equal to:

- ☐ 0
- ☐ πR^2
- ☐ $\frac{3}{4}\pi R$
- ☐ $\frac{3}{5}\pi^2 R$

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Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

F is conservative (*i.e.* it derives from a potential)

- ☐ over $\Omega = \{(x, y) : 1 \leq x \leq 3, 2 \leq y \leq 10\}$.
- ☐ over $\Omega = \{(x, y) : 2 \leq x^2 + y^2 \leq 4\}$.
- ☐ over $\Omega = \{(x, y) : x^2 + y^2 \leq 10\}$.
- ☐ for any domain Ω.

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Question 1 All the non-zero real Fourier coefficients (sine-cosine form) of the function f defined by:

$$f(x) = \underbrace{5}_{C_0} + \underbrace{\frac{1}{2}(5 - i\pi)e^{3ix}}_{C_3} + \underbrace{\frac{1}{2}(5 + i\pi)e^{-3ix}}_{C_{-3}} - \underbrace{\frac{3}{2}e^{7ix}}_{C_7} - \underbrace{\frac{3}{2}e^{-7ix}}_{C_{-7}}$$

are

- ☐ a_0, a_3, a_7
- ☐ a_3, a_7, b_3
- ☐ a_0, a_3, b_3, b_7
- ☐ $a_0,\ a_3,\ a_7,\ b_7$
- ☒ a_0, a_3, a_7, b_3

$$Ff(x) = \sum_{n=-\infty}^{+\infty} C_n\, e^{i\frac{2\pi n}{T}x} \qquad T = 2\pi$$

$$Ff(x) = \sum_{n=-\infty}^{+\infty} C_n{}^{inx}, \qquad C_0 = 5,\ C_3 = \frac{1}{2}(5 - i\pi)$$

$$C_{-3} = \frac{1}{2}(5 + i\pi),\quad C_7 = -\frac{3}{2},\quad C_{-7} = -\frac{3}{2}$$

$$C_0 = \frac{a_0}{2},\qquad \underbrace{C_n = \frac{a_n - ib_n}{2}}_{a_n = C_n + C_{-n}},\qquad \underbrace{C_{-n} = \frac{a_n + ib_n}{2}}_{b_n = i(C_n - C_{-n})}$$

$a_0 = 2C_0 = 10.$

a_0, a_3, b_3, a_7

$a_3 = C_3 + C_{-3} = 5 \neq 0$

$b_3 = i\left(\frac{1}{2}(5 - i\pi) - \frac{1}{2}(5 + i\pi)\right) = \pi \neq 0$

$a_7 = C_7 + C_{-7} = -3 \neq 0.$

$b_7 = i(C_7 - C_{-7}) = 0.$

Question 2 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\displaystyle\int_\Gamma f \, \mathrm{d}l$ is equal to:

- [ ] 0
- [x] $2\pi R$
- [ ] $2\pi R^2 + \pi R + 1$
- [ ] $2\pi R^3 + \pi R^2 + R$

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$$f(x,y) = xy + x + 1$$

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto R(\cos t, \sin t)$$

$$\gamma'(t) = R(-\sin t, \cos t), \quad \|\gamma'(t)\| = R$$

$$I = \int_\Gamma f \, dl = \int_0^{2\pi} f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

$$= \int_0^{2\pi} [\underbrace{R^2 \cos t \sin t}_{xy} + \underbrace{R \cos t}_{x} + 1] \underbrace{R}_{\|\gamma'(t)\|} \, dt$$

$$= R^3 \int_0^{2\pi} \cos t \sin t \, dt + \cancel{R^2 \int_0^{2\pi} \cos t} + R \int_0^{2\pi} dt$$

$$= \cancel{R^3 \frac{1}{2} \sin^2 t \Big|_0^{2\pi}} + R 2\pi = 2\pi R$$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \overset{(y,x)}{\cancel{(x,y)}},$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of A by ∂A, and the outer unit normal of ∂A by $\nu : \partial A \to \mathbb{R}^2$. The integral $\displaystyle\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- [x] 0
- [ ] πR^2
- [ ] $\frac{3}{4}\pi R$
- [ ] $\frac{3}{5}\pi^2 R$

$$\int_{\partial A} F \cdot \nu \, dl = \int_A \operatorname{div} F \, ds \quad \text{(Divergence theorem)}$$

$$\operatorname{div} F = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 \qquad \int_{\partial A} F \cdot \nu \, dl = 0$$

Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

F is conservative (*i.e.* it derives from a potential)

- ☐ over $\Omega = \{(x, y) : 1 \le x \le 3, 2 \le y \le 10\}$.
- ☐ over $\Omega = \{(x, y) : 2 \le x^2 + y^2 \le 4\}$.
- ☐ over $\Omega = \{(x, y) : x^2 + y^2 \le 10\}$.
- ☒ for any domain Ω.

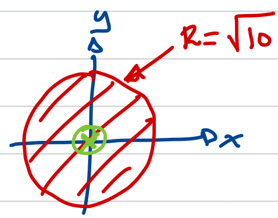
$\text{curl}\, F = 0?$, $\text{curl}\, F = \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}$

$$= \frac{1}{(x^2+y^2)^2}\left[\cancel{x^2}+\cancel{y^2}-\cancel{2x^2}+\cancel{x^2}+\cancel{y^2}-\cancel{2y^2}\right] = 0.$$

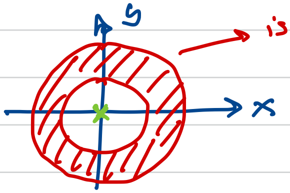
F is not defined at $(0,0)$

☐ ~~for any domain Ω.~~

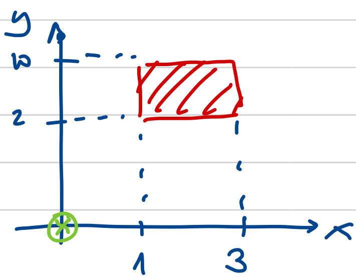
☐ ~~over $\Omega = \{(x, y) : x^2 + y^2 \le 10\}$.~~



☐ ~~over $\Omega = \{(x, y) : 2 \le x^2 + y^2 \le 4\}$.~~



is not simply connected



☒ over $\Omega = \{(x, y) : 1 \le x \le 3, 2 \le y \le 10\}$.

Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 4: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ■9

(i) Let Γ be the curve defined by

$$\Gamma = \left\{ \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \,\middle|\, t \in [-1, 1] \right\}.$$

Compute the length of Γ.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = (x^2, y\cos(x^2))$$

and Ω the triangle whose vertices are $(0, 0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu \mathrm{dl}$$

where $\nu : \partial\Omega \to \mathbb{R}^2$ is outer unit normal field of the boundary of Ω.

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i) $\boxed{\text{length}(\Gamma)} = \int_\Gamma 1 \, dl = \int_{-1}^{1} 1 \, \|\gamma'(t)\| \, dt$

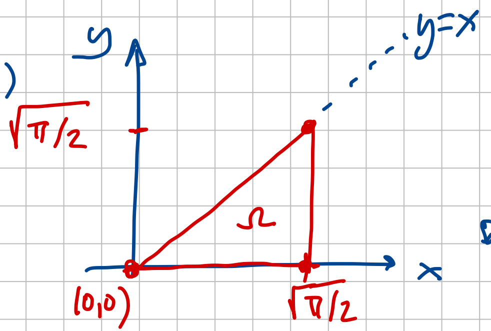
$\gamma'(t) = (t^2, 3, \sqrt{6}\, t)$, $\|\gamma'(t)\| = \sqrt{t^4 + 9 + 6t^2}$

$= \sqrt{(t^2+3)^2} = t^2 + 3$

$\text{length}(\Gamma) = \int_{-1}^{1} (t^2+3)\, dt = \left[\frac{1}{3}t^3 + 3t\right]_{-1}^{+1}$

$= \frac{1}{3} + 3 + \frac{1}{3} + 3 = 6 + \frac{2}{3} = \boxed{\frac{20}{3}}$

ii)



$\int_{\partial\Omega} F \cdot \nu \, dl = \int_\Omega \operatorname{div} F \, ds$

Diverg. theorem

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$$F = (x^2, y\cos(x^2)) \longrightarrow \operatorname{div} F = \underline{2x} + \underline{\cos(x^2)}$$
$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \text{ and } 0 \le x \le \sqrt{\frac{\pi}{2}}\}$$

$$\boxed{\iint_\Omega \operatorname{div} F \, ds} = \int_0^{\sqrt{\pi/2}} \int_0^x (2x + \cos(x^2))\, dy\, dx$$

$$= \int_0^{\sqrt{\pi/2}} (2xy + y\cos(x^2)) \Big|_0^x \, dx$$

$$= \int_0^{\sqrt{\pi/2}} (2x^2 + x\cos(x^2))\, dx = \frac{2}{3} x^3 \Big|_0^{\sqrt{\pi/2}} + \frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi/2}}$$

$$= \frac{2}{3}\left(\sqrt{\frac{\pi}{2}}\right)^3 + \frac{1}{2}\sin(\pi/2) = \boxed{\frac{\pi^{3/2}}{3\sqrt{2}} + \frac{1}{2}}$$

$$= \int_{\partial\Omega} F \cdot \nu \, dl$$

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Question 5: *This question is worth 6 points.*

0 1 2 3 4 5 6

Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

(i) Compute curl F.

(ii) Determine if F derives from a potential in Ω. If it does, find a potential of F, otherwise, justify why it does not derive from a potential in Ω.

i) $\boxed{\text{curl } F = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}}$

$$= \frac{1}{(x^2+y^2)^2}\left(x^2+y^2 - 2x(x+y) + x^2+y^2 + 2y(x-y) \right)$$

$$= \frac{1}{(x^2+y^2)^2}\left(\cancel{x^2} + \cancel{y^2} - \cancel{2x^2} - \cancel{2xy} + \cancel{x^2} + \cancel{y^2} + \cancel{2xy} - \cancel{2y^2} \right)$$

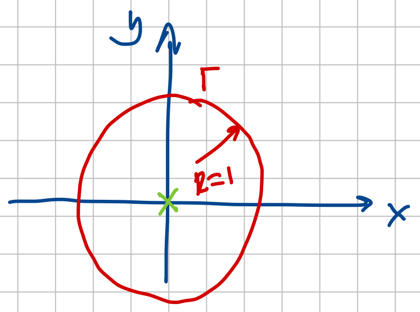
$$= \boxed{0}$$

ii) $\Omega = \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow$ nor convex nor simply connected

Applying the theorem $\longrightarrow$ we don't know if F derives from a potential on Ω.

If $\int_\Gamma F \cdot d\ell \neq 0 \longrightarrow F$ does not derive from a potential

$\Gamma \subset \Omega$ is a closed curve.



$\Gamma = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$t \longmapsto (\cos t, \sin t)$

$\gamma'(t) = (-\sin t, \cos t)$

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$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt =$$

$$= \int_0^{2\pi} (\cos t - \sin t,\ \cos t + \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (\cancel{-\sin t \cos t} + \sin^2 t + \cos^2 t + \cancel{\sin t \cos t})\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

$$= \int_\Gamma F \cdot dl \neq 0 \longrightarrow$$ F does not derive from a potential in Ω

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Question 6: *This question is worth 3 points.*

☐ 0 ☐ 1 ☐ 2 ■ 3

Let $F : \mathbb{R}^2 \to \mathbb{R}^2; F(x,y) = (F_1(x,y), F_2(x,y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and div $F = 0$. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by:

$$G(x,y) = (F_2(-x,y), F_1(-x,y)).$$

Show that G derives from a potential in $\mathbb{R}^2$.

The domain is $\Omega = \mathbb{R}^2$ is convex and simply connected.

If $\operatorname{curl} G = 0 \longrightarrow$ then, G derives from a potential in Ω.

$\operatorname{curl} G = 0$?

$$\boxed{\operatorname{curl} G} = \frac{\partial G_y(x,y)}{\partial x} - \frac{\partial G_x(x,y)}{\partial y}$$

$$= \frac{\partial F_1(t,y)}{\partial x} - \frac{\partial F_2(t,y)}{\partial x} \quad \text{where } t = -x. \quad t(x) = -x$$

$$= \frac{\partial F_1(t,y)}{\partial t} \underbrace{\frac{\partial t}{\partial x}}_{=-1} - \frac{\partial F_2(t,y)}{\partial y}$$

$$= -\left(\frac{\partial F_1(t,y)}{\partial t} + \frac{\partial F_2(t,y)}{\partial y} \right) = -\operatorname{div} F = \boxed{0}$$

G derives from a potential in Ω.

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Question 7: *This question is worth 14 points.*

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☐8 ☐9 ☐10 ☐11 ☐12 ☐13 ■14

Let $F:\mathbb{R}^3\to\mathbb{R}^3$ be the vector field defined as $F(x,y,z)=(0,x,0)$ and let Σ be the surface defined by

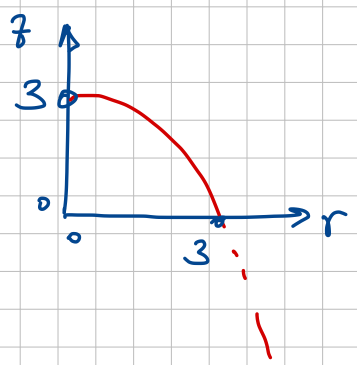
$$\Sigma=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; z=\left(\sqrt{\underbrace{x^2+y^2}_{r^2}}+1\right)\left(3-\sqrt{\underbrace{x^2+y^2}_{r^2}}\right), y\geq 0, z\geq 0\right\}.$$

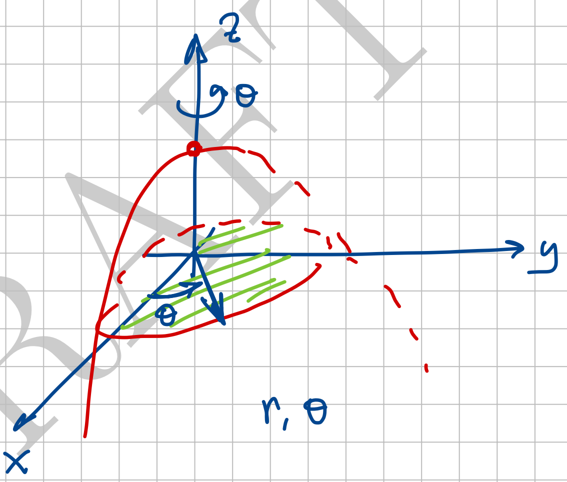
Verify the Stokes theorem for F and Σ.
Note: if necessary, use the following formulas:

$$\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$$

$$\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$$

$r=\sqrt{x^2+y^2}$, $z(r)=(r+1)(3-r)$





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$y\geq 0 \to \theta\in[0,\pi]$

$z\geq 0 \to r\in[0,3]$

$\sigma:[0,3]\times[0,\pi]\longrightarrow\mathbb{R}^3$

$(r,\theta)\longmapsto(r\cos\theta,\ r\sin\theta,\ (r+1)(3-r))$

$\frac{\partial\sigma}{\partial r}=\sigma_r=(\cos\theta,\sin\theta,\ 2-2r)$

$\frac{\partial\sigma}{\partial\theta}=\sigma_\theta=(-r\sin\theta,\ r\cos\theta,\ 0)$

$\sigma_r\wedge\sigma_\theta=r\begin{pmatrix}(2r-2)\cos\theta\\(2r-2)\sin\theta\\1\end{pmatrix}$

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Stokes theorem:

$$\iint_{\Sigma} \text{curl}\, F \cdot ds = \int_{\partial\Sigma} F \cdot dl$$

$$\text{curl}\, F = \begin{array}{c} \\ \\ F_x \quad F_y \quad F_z \longrightarrow \end{array} \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

left-hand side:

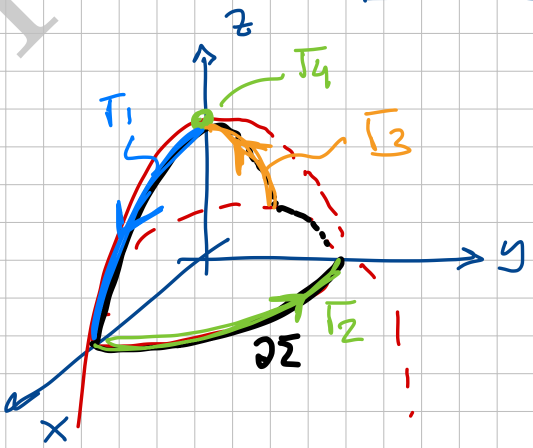
$$\boxed{\iint_{\Sigma} \text{curl}\, F \cdot ds} = \int_0^3 \int_0^{\pi} (\text{curl}\, F)(\sigma(r,\theta)) \cdot \sigma_r \wedge \sigma_\theta \, d\theta\, dr$$

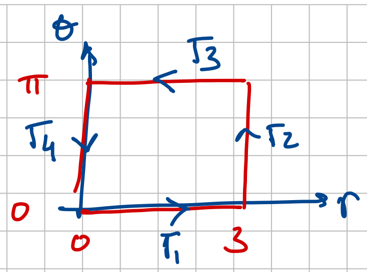
$$= \int_0^3 \int_0^{\pi} r\,(0,0,1) \cdot \big((2r-2)\cos\theta, (2r-2)\sin\theta, 1 \big)\, d\theta\, dr$$

$$= \int_0^3 \int_0^{\pi} r\, d\theta\, dr = \pi \int_0^3 r\, dr = \frac{\pi}{2} r^2 \Big|_0^3 = \boxed{\frac{9}{2}\pi}$$

Right-hand-side:

$$\int_{\partial\Sigma} F \cdot dl$$





$\Gamma_1 = \{ \gamma_1(r) = \sigma(r,0) = (r, 0, (r+1)(3-r)) \text{ with } r: 0 \to 3 \}$

$\Gamma_2 = \{ \gamma_2(\theta) = \sigma(3,\theta) = (3\cos\theta, 3\sin\theta, 0) \text{ with } \theta: 0 \to \pi \}$

$\Gamma_3 = \{ \gamma_3(r) = \sigma(r,\pi) = (-r, 0, (r+1)(3-r)) \text{ with } r: 3 \to 0 \}$

$\Gamma_4 = \{ \gamma_4(\theta) = \sigma(0,\theta) = \underbrace{(0,0,3)}_{\text{single point.}} \text{ with } \theta: \pi \to 0 \}$

$\partial\Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$$\int_{\partial\Sigma} F \cdot dl = \int_{\Gamma_1} F \cdot dl + \int_{\Gamma_2} F \cdot dl + \int_{\Gamma_3} F \cdot dl$$

$\gamma_1'(r) = (1, 0, 2-r)$, $\gamma_2'(\theta) = 3(-\sin\theta, \cos\theta, 0)$

$\gamma_3'(r) = (-1, 0, 2-r)$

$$\int_{\Gamma_1} F \cdot dl = \int_0^3 \underbrace{(0,r,0)}_{F(\gamma_1(r))} \cdot \underbrace{(1,0,2-r)}_{\gamma_1'(r)} dr = 0$$

$$\int_{\Gamma_3} F \cdot dl = \int_3^0 \overbrace{(0,-r,0)} \cdot \overbrace{(-1, 0, 2-r)} dr = 0$$

$$\int_{\Gamma_2} F \cdot dl = \int_0^\pi \underbrace{3(0,\cos\theta,0)}_{F(\gamma_2(\theta))} \cdot \underbrace{3(-\sin\theta, \cos\theta, 0)}_{\gamma_2'(\theta)} d\theta$$

$$= 9 \int_0^\pi \cos^2\theta \, d\theta = \frac{9}{2} \int_0^\pi (1 + \cos 2\theta) d\theta = \frac{9\pi}{2} + \overbrace{\int_0^\pi \cos 2\theta \, d\theta}^{=0} = \boxed{\frac{9\pi}{2}}$$

For your examination, preferably print documents compiled from auto-multiple-choice.

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Question 8: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ■9

Let $f : [0, \pi] \to \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

(i) Compute $F_s f$, the Fourier series in sines of f.

(ii) Using the course's results, compare $F_s f$ and f in the interval $[0, \pi]$.

i) $F_s f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{\pi} x\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\sin(nx)\,dx$$

$u = 2\pi x - x^2$
$dv = \sin(nx)dx$
$v = -\frac{1}{n}\cos(nx)$

$$= \frac{2}{\pi n}\left[-(2\pi x - x^2)\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}(2\pi - 2x)\cos(nx)\,dx\right]$$

$$= -\frac{2\pi}{n}(-1)^n + \frac{4}{\pi n}\int_0^{\pi}(\pi - x)\cos(nx)\,dx$$

$$= \frac{2\pi}{n}(-1)^{n+1} + \underbrace{\frac{4}{\pi n}\int_0^{\pi}\pi\cos(nx)\,dx}_{=0.} - \frac{4}{\pi n}\int_0^{\pi}x\cos(nx)\,dx$$

$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n}\int_0^{\pi}x\cos(nx)\,dx$$

$u = x$
$dv = \cos(nx)dx$
$v = \frac{1}{n}\sin(nx)$

$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n}\left[\overbrace{\frac{x}{n}\sin(nx)\Big|_0^{\pi}}^{=0} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

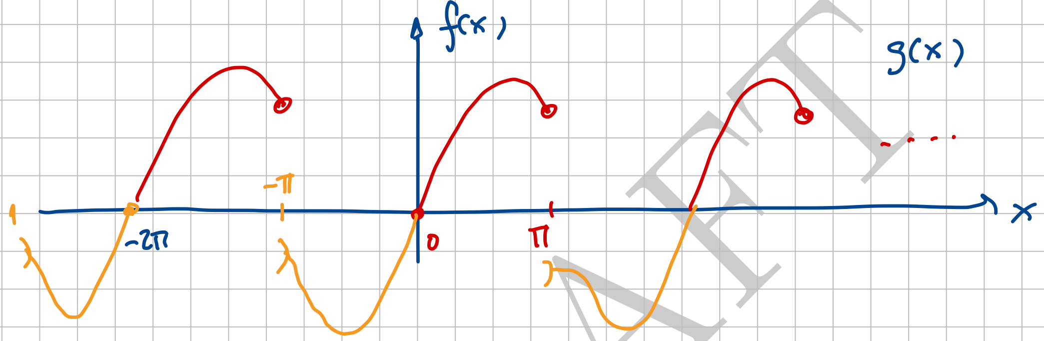
$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n^3}\cos(nx)\Big|_0^{\pi} = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}(-1)^{n+1} + \frac{4}{\pi n^3}$$

$$= \left(\frac{2\pi}{n} + \frac{4}{\pi n^3}\right)(-1)^{n+1} + \frac{4}{\pi n^3}$$

$$F_s f(x) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^3} + \frac{2\pi^2 n^2 + 4}{\pi n^3} (-1)^{n+1} \right] \sin(nx).$$

ii) when $f(0) = f(\pi) = 0 \rightarrow f(x) = F_s f(x)$.
f is continuous

$f(x) = -x^2 + 2\pi x, \quad f(0) = 0, \quad f(\pi) = \pi^2 \neq 0.$



$F_s f(x) = F g(x)$, $g(x)$ is a piecewise-defined function.

I apply Dirichlet theorem to $g(x)$ on $[0, \pi]$

- $F g(x) = F_s f(x) = g(x) = f(x)$ on $[0, \pi[$

- However on $x = \pi$, $g(x)$ is discontinuous and we can only say that

$$F_s f(\pi) = F g(\pi) = \frac{1}{2}\left(g(\pi+0) + g(\pi-0)\right)$$

$$= \frac{1}{2}\left(-\pi^2 + \pi^2\right) = 0 \neq f(\pi) = g(\pi) = \pi^2$$

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Question 9: *This question is worth 4 points.*

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Let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x & \text{if} \quad -\pi \leq x \leq 0 \\ \pi & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of g are

$$a_0 = \frac{3\pi}{2};$$
$$a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases} \qquad \text{for } n \geq 1;$$
$$b_n = \frac{1}{n} \qquad \text{for } n \geq 1.$$

Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{(k-\frac{1}{2})^2}.$$

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$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$T = 2\pi$

$$= \frac{3\pi}{4} - \sum_{n \text{ odd}} \frac{2}{n^2\pi} \cos(nx) + \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

$$= \frac{3\pi}{4} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi} \cos((2n+1)x) + \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$$

$$Ff(0) = \frac{3\pi}{4} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi} \underbrace{\cos(0)}_{=1} + \sum_{n=1}^{\infty} \frac{1}{n} \underbrace{\sin(0)}_{=0}$$

$$= \frac{3\pi}{4} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi} = \frac{3\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$= \frac{3\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2} \frac{1}{4}$$

$k = n+1$

$$= \frac{3\pi}{4} - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2}$$

Dirichlet theorem:

$$Ff(0) = \frac{1}{2} f(0^+) + \frac{1}{2} f(0^-) = \frac{1}{2} f(0+0) + \frac{1}{2} f(0-0)$$

$$= \frac{1}{2}(0+\pi) = \frac{\pi}{2}$$

$$\frac{3\pi}{4} - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2} = \frac{\pi}{2}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2} = \frac{\pi^2}{2}}$$

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Question 10: *This question is worth 8 points.*

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(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \to \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} \Big(9u(t) - 4u''(t)\Big) e^{-\frac{3}{2}|x-t|} dt = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

i) let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise-defined function s.t. $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ then:

$$\hat{f} : \mathbb{R} \to \mathbb{C}$$
$$\alpha \mapsto \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

ii) $$-10u(x) + \int_{-\infty}^{+\infty} \underbrace{(9u(t) - 4u''(t))}_{f(x)} e^{-3/2|x-t|} dt$$

$$= \underbrace{\frac{4x^2}{(2\pi + x^2)^2}}_{h(x)}$$

$$-10u(x) + f(x) * g(x) = h(x)$$

$$g(x) = e^{-3/2|x|}$$

Apply Fourier transform:

$$-10\hat{u}(\alpha) + \sqrt{2\pi}\,\hat{f}(\alpha)\hat{g}(\alpha) = \hat{h}(\alpha)$$

$$\hat{f}(\alpha) = 9\hat{u}(\alpha) - 4(i\alpha)^2\hat{u}(\alpha)$$

$$\hat{g}(\alpha) = \frac{3}{2}\sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2 + 9/4}$$

Table row 7

$\omega = 3/2$

$$-10\hat{u}(\alpha) + \sqrt{2\pi}\left(9\hat{u}(\alpha) + 4\alpha^2\hat{u}(\alpha)\right)\hat{g}(\alpha) = \hat{h}(\alpha)$$

$$\hat{u}(\alpha)\left[\sqrt{2\pi}(9 + 4\alpha^2)\hat{g}(\alpha) - 10\right] = \hat{h}(\alpha)$$

$$\hat{u}(\alpha)\left[\sqrt{2\pi}(9+4\alpha^2)\underbrace{\frac{3}{2}\sqrt{\frac{2}{\pi}}\frac{4}{9+\alpha^2}}_{\hat{g}(\alpha)} - 10\right] = \hat{h}(\alpha)$$

$$\hat{u}(\alpha)[12 - 10] = \hat{h}(\alpha) \longrightarrow \hat{u}(\alpha) = \frac{1}{2}\hat{h}(\alpha)$$

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Apply the inverse Fourier transform.

$$\mathcal{F}^{-1}(\hat{u}(\alpha))(x) = \frac{1}{2}\mathcal{F}^{-1}(\hat{h}(\alpha))(x)$$

$$u(x) = \frac{1}{2}h(x) \Rightarrow \boxed{u(x) = \frac{2x^2}{(2\pi + x^2)^2}}$$

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