



Teacher : Teachers of Analysis III
Analysis III - Mock exam - Student
December 2021
Duration : 120 minutes

Student One

SCIPER: 111111

Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
 - +2 points if your answer is correct,
 - 0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes Observe this guidelines Beachten Sie bitte die unten stehenden Richtlinien		
choisir une réponse select an answer Antwort auswählen	ne PAS choisir une réponse NOT select an answer NICHT Antwort auswählen	Corriger une réponse Correct an answer Antwort korrigieren
  		 
ce qu'il ne faut PAS faire what should NOT be done was man NICHT tun sollte		
     		

**First part: multiple choice questions**

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 All the non-zero real Fourier coefficients (sine-cosine form) of the function f defined by:

$$f(x) = 5 + \frac{1}{2}(5 - i\pi)e^{3ix} + \frac{1}{2}(5 + i\pi)e^{-3ix} - \frac{3}{2}e^{7ix} - \frac{3}{2}e^{-7ix}$$

are

- ☐ a_0, a_3, a_7
☐ a_3, a_7, b_3
☐ a_0, a_3, b_3, b_7
☐ a_0, a_3, a_7, b_7

Question 2 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}; (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curve defined by:

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\}.$$

The integral $\int_{\Gamma} f \, dl$ is equal to:

- ☐ 0
☐ $2\pi R$
☐ $2\pi R^2 + \pi R + 1$
☐ $2\pi R^3 + \pi R^2 + R$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2; (x, y) \mapsto (x, y),$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}.$$

We also denote the boundary of A by ∂A , and the outer unit normal of ∂A by $\nu : \partial A \rightarrow \mathbb{R}^2$.

The integral $\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- ☐ 0
☐ πR^2
☐ $\frac{3}{4}\pi R$
☐ $\frac{3}{5}\pi^2 R$



Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2; (x, y) \mapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

F is conservative (*i.e.* it derives from a potential)

- ☐ over $\Omega = \{(x, y) : 1 \leq x \leq 3, 2 \leq y \leq 10\}$.
- ☐ over $\Omega = \{(x, y) : 2 \leq x^2 + y^2 \leq 4\}$.
- ☐ over $\Omega = \{(x, y) : x^2 + y^2 \leq 10\}$.
- ☐ for any domain Ω .

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Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 4: *This question is worth 9 points.*

☐	0	☐	1	☐	2	☐	3	☐	4	☐	5	☐	6	☐	7	☐	8	☒	9
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- (i) Let Γ be the curve defined by

$$\Gamma = \left\{ \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \mid t \in [-1, 1] \right\}.$$

Compute the length of Γ .

- (ii) Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector field defined by

$$F(x, y) = (x^2, y \cos(x^2))$$

and Ω the triangle whose vertices are $(0, 0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu dl$$

where $\nu : \partial\Omega \rightarrow \mathbb{R}^2$ is outer unit normal field of the boundary of Ω .





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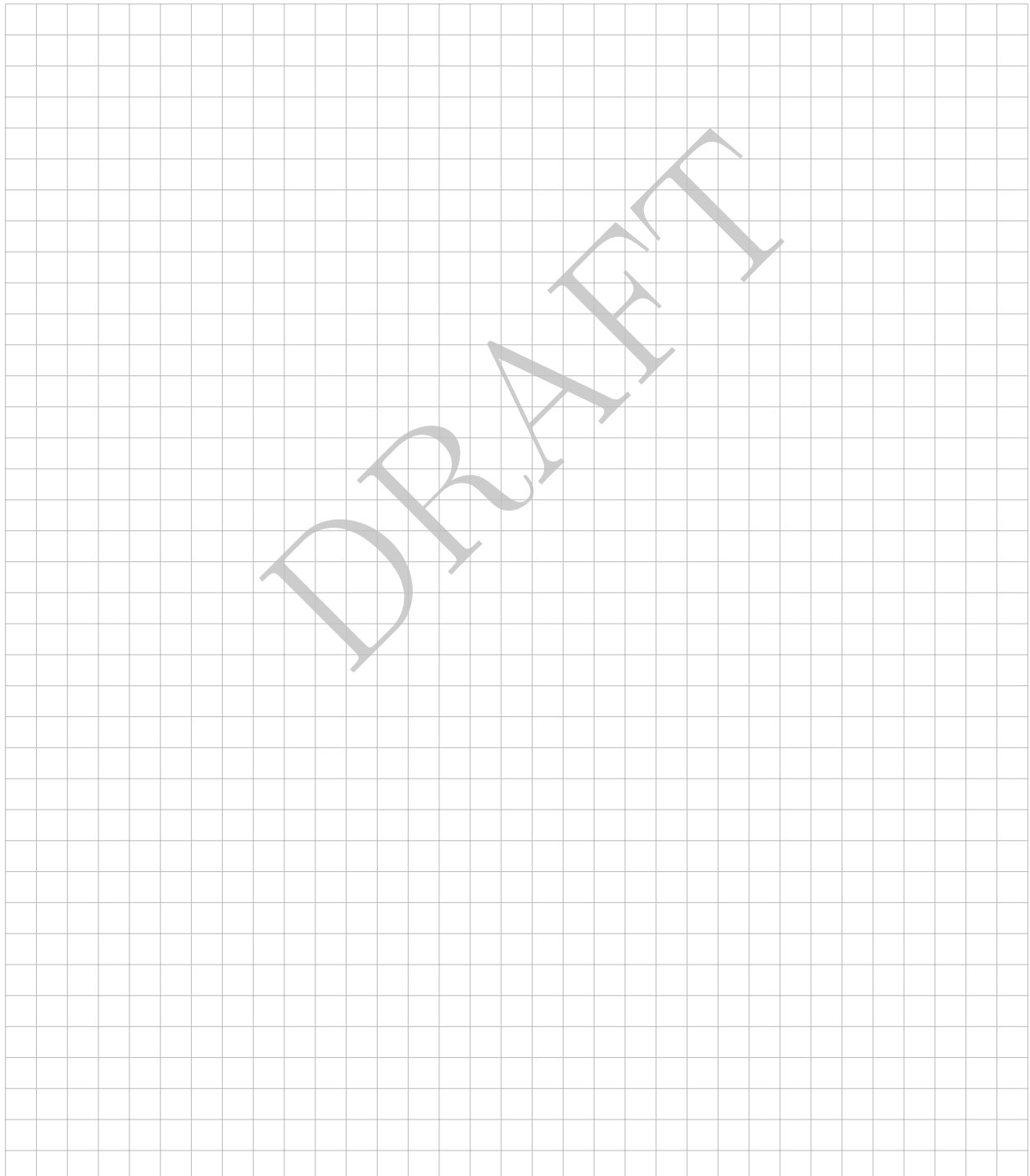
Question 5: *This question is worth 6 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☒ 6

Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$F(x, y) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

- (i) Compute $\text{curl } F$.
- (ii) Determine if F derives from a potential in Ω . If it does, find a potential of F , otherwise, justify why it does not derive from a potential in Ω .





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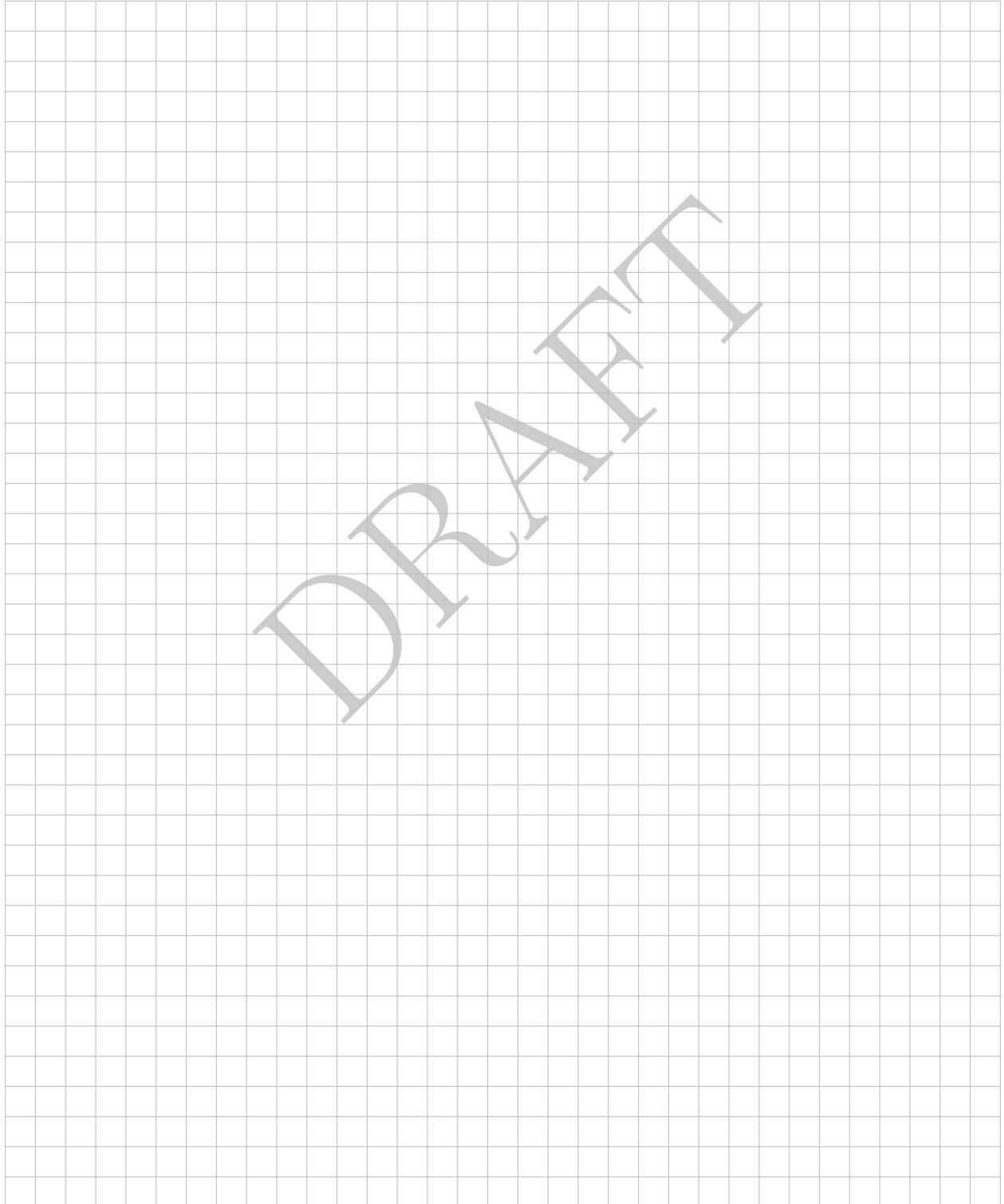
Question 6: *This question is worth 3 points.*

☐ 0 ☐ 1 ☐ 2 ☒ 3

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2; F(x, y) = (F_1(x, y), F_2(x, y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\operatorname{div} F = 0$.
Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field defined by:

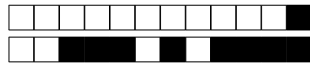
$$G(x, y) = (F_2(-x, y), F_1(-x, y)).$$

Show that G derives from a potential in $\mathbb{R}^2$.





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Question 7: *This question is worth 14 points.*

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☐	8	☐	9	☐	10	☐	11	☐	12	☐	13	☒	14		

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field defined as $F(x, y, z) = (0, x, 0)$ and let Σ be the surface defined by

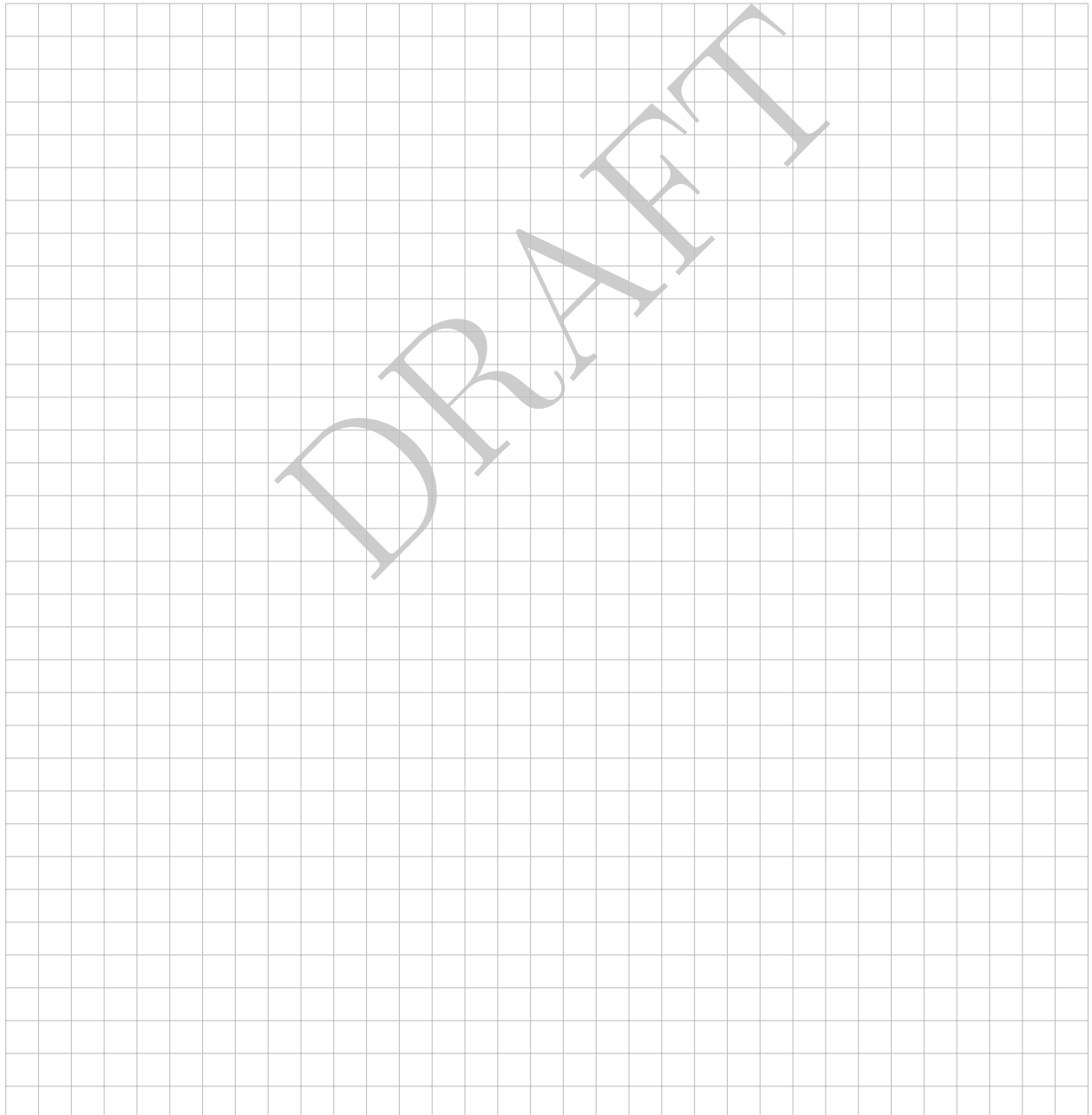
$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = \left(\sqrt{x^2 + y^2} + 1 \right) \left(3 - \sqrt{x^2 + y^2} \right), y \geq 0, z \geq 0 \right\}.$$

Verify the Stokes theorem for F and Σ .

Note: if necessary, use the following formulas:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$$



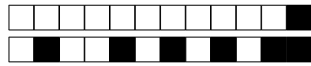


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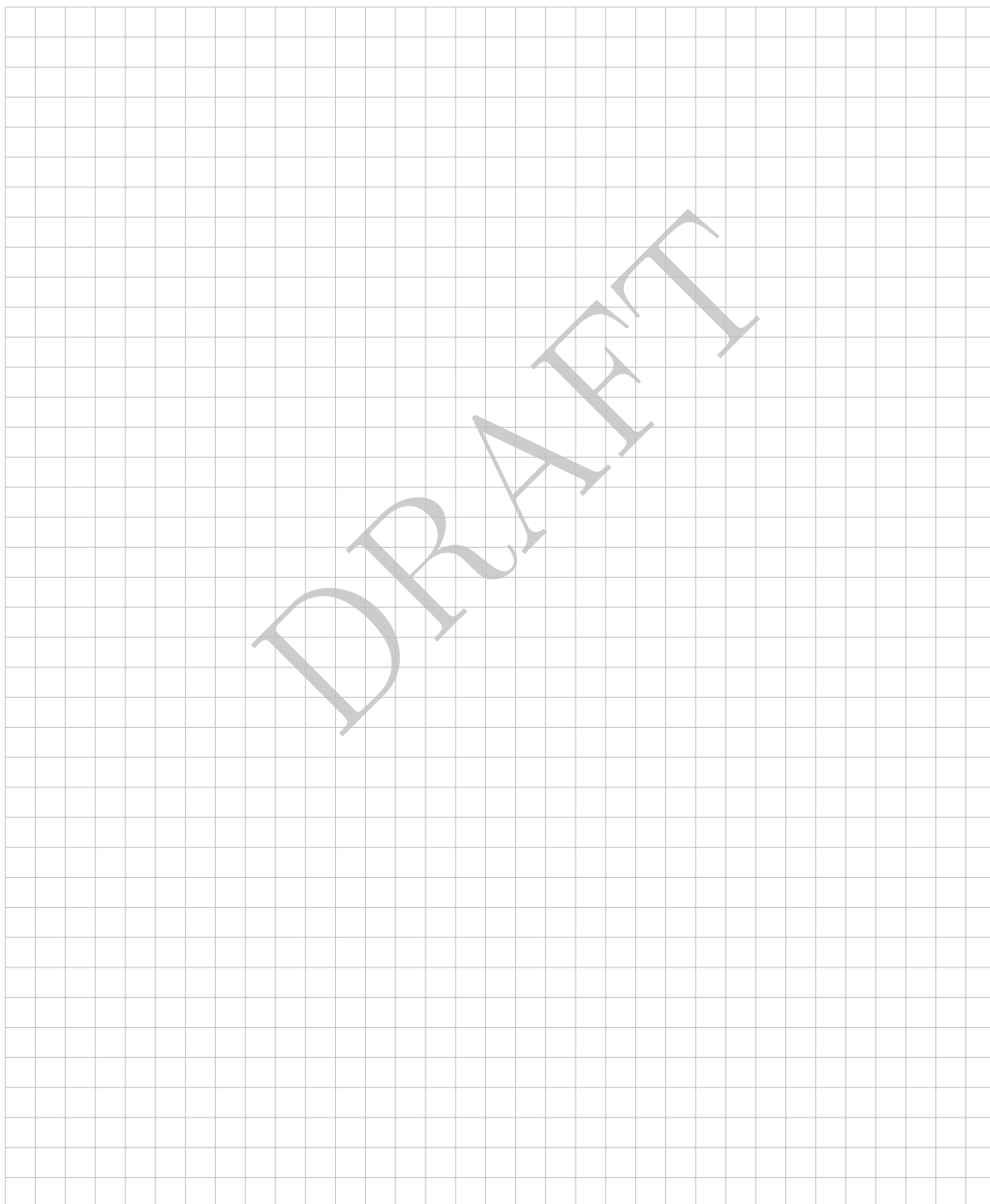


Question 8: *This question is worth 9 points.*

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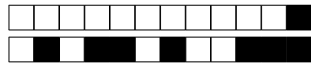
Let $f : [0, \pi] \rightarrow \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

- (i) Compute $F_s f$, the Fourier series in sines of f .
- (ii) Using the course's results, compare $F_s f$ and f in the interval $[0, \pi]$.





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Question 9: *This question is worth 4 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☒ 4

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x & \text{if } -\pi \leq x \leq 0 \\ \pi & \text{if } 0 < x < \pi \end{cases} \quad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of g are

$$\begin{aligned} a_0 &= \frac{3\pi}{2}; \\ a_n &= \begin{cases} -\frac{2}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geq 1; \\ b_n &= \frac{1}{n} \quad \text{for } n \geq 1. \end{aligned}$$

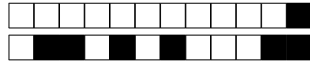
Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2}.$$





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Question 10: This question is worth 8 points.

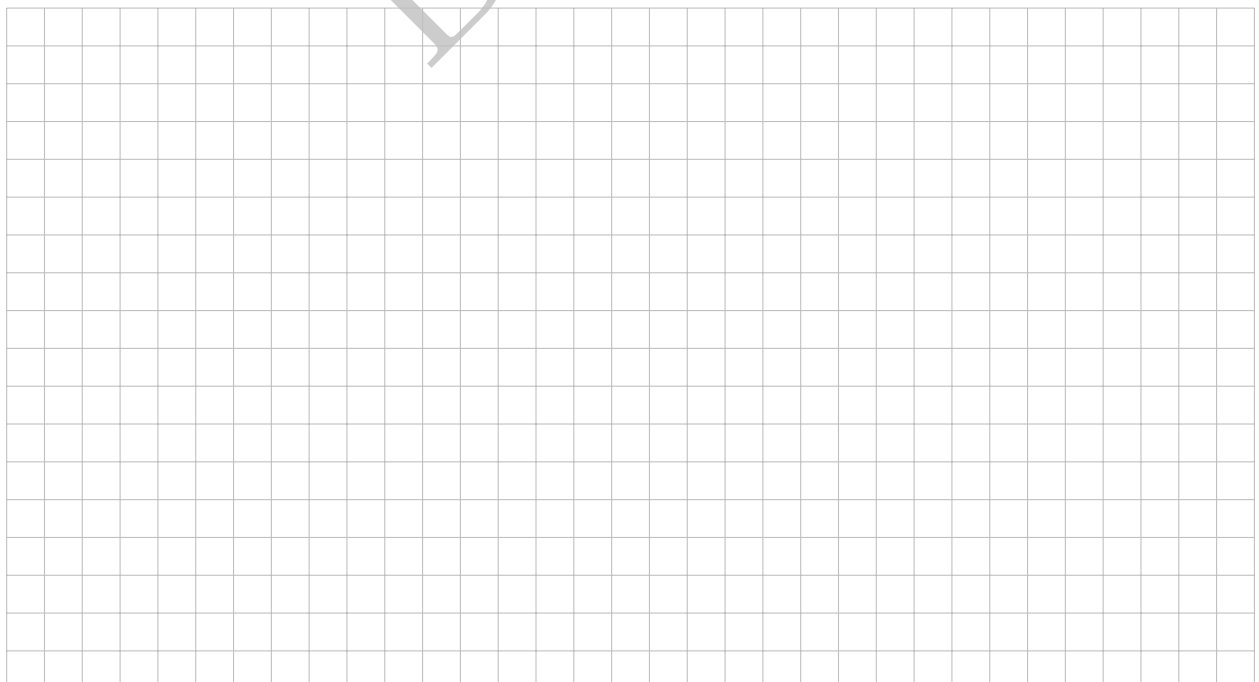
☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☒ 8

- (i) Write the definition of the Fourier transform of a function detailing its hypotheses
- (ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \rightarrow \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} (9u(t) - 4u''(t)) e^{-\frac{3}{2}|x-t|} dt = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } y < b \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(b \alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{- w\alpha }}{ w }$
7	$f(y) = \frac{e^{- wy }}{ w } \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2} w } e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2} w ^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{ w } - \alpha \right) e^{- w\alpha }$





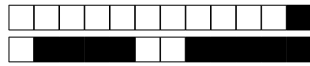
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