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Review of Fourier Analysis and applications

- Fourier series:

Idea: we want to expand a periodic function in orthonormal basis of

$$\left\{ \sin\left(kx \cdot \frac{2\pi}{T}\right) ; \cos\left(kx \cdot \frac{2\pi}{T}\right) \right\}.$$

So we define Fourier series of f to be

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right) \right]$$

where $a_n = \left\langle f, \cos\left(\frac{2\pi}{T} nx\right) \right\rangle = \frac{2}{T} \int_0^T f \cdot \cos\left(\frac{2\pi}{T} nx\right) dx$

$$b_n = \left\langle f, \sin\left(\frac{2\pi}{T} nx\right) \right\rangle = \frac{2}{T} \int_0^T f \cdot \sin\left(\frac{2\pi}{T} nx\right) dx$$

This idea makes sense by <u>Dirichlet theorem</u>:

Let $f: \mathbb{R} \to \mathbb{R}$ be T-periodic, piecewise continuous function. then for every $x \in \mathbb{R}$

$Ff(x)$ converges and

$$\boxed{Ff(x) = \lim_{t \to 0} \frac{f(x+t) + f(x-t)}{2}}$$

If $f(x)$ is continuous then $Ff(x) = f(x)$ $\forall x \in \mathbb{R}$

<u>Example</u> Let $f(x) = \begin{cases} x & 0 \leq x < 1 \\ \cos(x) & 1 \leq x < 2 \end{cases}$ ~~and~~ extended by 2-periodicity

then $Ff(x) = f(x)$ for $x \in [0,2)$ and $x \neq 0, 1$

for & $Ff(0) = \dfrac{0 + \cos(2)}{2} = \dfrac{\cos(2)}{2}$

$Ff(1) = \dfrac{1 + \cos(1)}{2}$.

We can use variations of Fourier series.

Corollary 14.4.

- If f is even periodic function, we get:

$$Ff(x) = \frac{a_0}{2} + \sum_{n \geq 1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right)$$

- If f is odd periodic function, we get:

$$Ff(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right)$$

- We have complex Fourier coefficients:

$$\boxed{Ff(x) = \sum_{n=-\infty}^{n=+\infty} c_n \cdot e^{i \frac{2\pi n}{T} x} \quad \text{with} \quad c_n = \frac{1}{T} \int_0^T f(x)\, e^{-i \frac{2\pi n}{T} x}\, dx.}$$

We can relate & complex coeff. to real via:

$\forall n \geq 1$ $c_n = \dfrac{a_n - i b_n}{2}$ for $n = 0$ $c_0 = \dfrac{a_0}{2}$ for $n \leq -1$ $c_n = \dfrac{a_n + i b_n}{2}$

We can differentiate and integrate Fourier series term by term:

Theorems 14.5, 14.6

$$Ff(x)' = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right]'$$

$$= \sum_{n=1}^{\infty} \frac{2\pi n}{T} \left(b_n \cos \frac{2\pi n}{T} x - a_n \sin \frac{2\pi n}{T} x \right) =$$

$$= \lim_{t \to 0} \frac{f'(x+t) + f'(x-t)}{2}$$

$$\int_{x_0}^{x} f(t)\, dt = \int_{x_0}^{x} \frac{a_0}{2}\, dt +$$

$$+ \sum_{n=1}^{\infty} \int_{x_0}^{x} a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) dt$$

this can be used to solve ODEs

Chapter 17.3

Example ~~Find~~ Ex. Let f ~~be~~ $\in C^1(\mathbb{R})$

be 2π- periodic. For $m, k \in \mathbb{R}$ with $m \neq 0$

find $u(t)$ s.t.

$$\begin{cases} m\,u''(t) + k\,u(t) = f(t) & t \in (0, 2\pi) \\ u(0) = u(2\pi), & y'(0) = y'(2\pi) \end{cases}$$

Solution Lets ~~ex~~ write both sides as Fourier expansions:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

$$u(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nt) + B_n \sin(nt)$$

We will solve equations for A_n, B_n:

$$m\cdot u''(t)+k\,u(t)=\frac{kA_0}{2}+\sum_{n=1}^{\infty}(k-mn^2)A_n\cos(nt)+(k-mn^2)B_n\sin(nt).$$

$$\|$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nt)+b_n\sin(nt)$$

So we get $\frac{kA_0}{2}=\frac{a_0}{2}\Rightarrow \boxed{A_0=\frac{a_0}{k}}$

$$(k-mn^2)A_n=a_n\Rightarrow \boxed{A_n=\frac{a_n}{k-mn^2}}$$

$$(k-mn^2)B_n=b_n= \boxed{B_n=\frac{b_n}{k-mn^2}}$$

Example: Let $f\in C^1(\mathbb{R})$ be 2π-periodic and $\alpha\neq\pm 1$. find function $u(t)$ s.t.

$$\begin{cases} u(t)+\alpha\, u(t-\pi)=f(t) \\ u(0)=u(2\pi) \end{cases}$$

Solution: Again we use Fourier expansion:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$u(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nt) + B_n \sin(nt)$$

$$u(t-\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n(t-\pi)) + B_n \sin(n(t-\pi))$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} (-1)^n A_n \cos(nt) + (-1)^n B_n \sin(nt).$$

$$\Rightarrow \quad u(t) + \alpha \cdot u(t-\pi) =$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} (1+\alpha \cdot (-1)^n) \cdot A_n \cos(nt) + (1+\alpha \cdot (-1)^n) B_n \sin(nt)$$

And we get

$$\boxed{A_0 = \frac{a_0}{1+\alpha} \qquad A_n = \frac{a_n}{1+\alpha(-1)^n} \qquad B_n = \frac{b_n}{1+\alpha(-1)^n}}$$

Another ~~the~~ important theorem is ~~the~~ Perseval identity:

$$\frac{2}{T}\int_0^T (f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Fourier ~~series~~. Transform

Definition $f: \mathbb{R} \to \mathbb{R}$ s.t. $\int_{-\infty}^{+\infty} |f|\,dx < \infty$

then $\mathcal{F}f(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha y}\, dy$

Properties of Fourier transform

Theorem 15.7.

- Linearity $\mathcal{F}(af + bg) = a\,\mathcal{F}f + b \cdot \mathcal{F}g$
- $\mathcal{F}(f')(\alpha) = i\alpha\, \mathcal{F}(f)(\alpha)$
- $\mathcal{F}(xf) = i \cdot \frac{d}{d\alpha} \mathcal{F}f(\alpha)$
- convolution $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$

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Plancherel identity:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} |\hat{f}(\alpha)|^2 \, d\alpha .$$

Example: Gaussian Function

$$f(x) = e^{-x^2}$$

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \cdot e^{-i\alpha x} \, dx =$$

$$= \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x+i\alpha)^2}{2}} \, dx = \quad \text{let } u = \frac{x+i\alpha}{\sqrt{2}}$$

$$= \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{\pi}} \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} \, du}_{\text{denote this by } I} = I \cdot \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{\pi}} .$$

By Plancherel:

$$\int_{-\infty}^{+\infty} |f|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}|^2 \, d\alpha$$

$$\int_{-\infty}^{+\infty} \left(e^{-\frac{x^2}{2}}\right)^2 dx = \int_{-\infty}^{+\infty} \left(\frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{\pi}} \cdot I\right)^2 d\alpha$$

$$\underbrace{\int_{-\infty}^{+\infty} e^{-x^2} dx}_{I} \qquad = \qquad \frac{I^2}{\pi} \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-\alpha^2} d\alpha}_{I}$$

$$\Rightarrow \quad I = \frac{I^3}{\pi} \Rightarrow \boxed{I = \sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} dx}$$

We also have inversion formula

Thm 15.3 (i) Let $\int_{-\infty}^{+\infty} |f| \, dx < \infty$, $\int_{-\infty}^{+\infty} |\hat{f}(\alpha)| \, d\alpha < \infty$

Let f be continuous then:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) \, e^{i\alpha x} \, d\alpha}$$

Also for even and odd functions we have cosine and sine fourier transforms:

Theorem 15.3 (ii)
If f is even:

$$\mathcal{F}f(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y) \cdot \cos(\alpha y) \, dy$$

(iii) If f is odd:

$$\mathcal{F}f(\alpha) = -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(y) \sin(\alpha y) \, dy$$

Application of Fourier transform

<u>Example</u> 17.7 find $u(t)$ s.t.

$$u(t) + 3\int_{-\infty}^{+\infty} e^{-|\tau|} \cdot u(t-\tau)\, d\tau = e^{-|t|}$$

<u>Solution:</u>

Note that: $\displaystyle\int_{-\infty}^{+\infty} e^{-|\tau|} \cdot u(t-\tau)\, dt =$

$$= e^{-|\tau|} * u(t)$$

then can apply Fourier transform on both sides:

$$\hat{u}(\alpha) + 3\sqrt{2\pi}\, \hat{u} \cdot F(e^{-|t|}) = F(e^{-|t|}) \Rightarrow$$

$$\Rightarrow \hat{u}(\alpha) = \frac{F(e^{-|t|})}{1 + 3\sqrt{2\pi}\, F(e^{-|t|})}$$

Note that $F(e^{-|t|}) = \dfrac{1}{\sqrt{2\pi}} \dfrac{2}{1+\alpha^2}$ so we get:

$$\hat{u}(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha^2 + 7}$$

Using properties of Fourier transform we get:

$$u(x) = F^{-1}\left(\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha^2+7}\right) = \frac{1}{\sqrt{7}} e^{-\sqrt{7}|t|}$$