20/11/2024

2.2.2 Fourier inversion theorem (reciprocity)

let $f: \mathbb{R} \to \mathbb{R}$ be s.t. f and f' are piecewise-defined

and $\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$ (i.e., there exist $\hat{f}$)

and $\int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha < \infty$

we have:

$$\mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha = \frac{1}{2}\left(f(x+0) + f(x-0) \right)$$

In the particular case in which f is continuous in x we have:

$\frac{1}{2}\left(f(x+0) + f(x-0) \right) = f(x)$, and then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha x} \, d\alpha$$

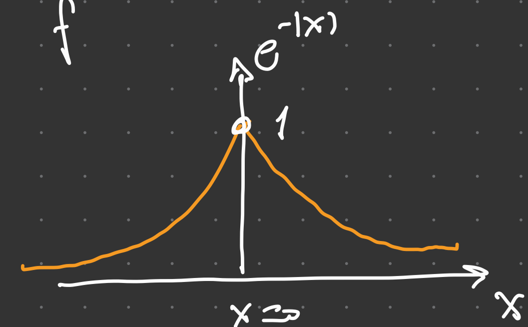
In other words $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$.

$$f \xrightarrow{\mathcal{F}} \hat{f} \xrightarrow{\mathcal{F}^{-1}} f \quad (\text{for } f \text{ continuous at } x).$$

2.2.3 Examples:

- Example 1: Compute the Fourier transform of f

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} |f(x)|\,dx \overset{?}{<} \infty$$

$$\int_{-\infty}^{+\infty} |e^{-|x|}|\,dx = \int_{-\infty}^{+\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{+\infty} e^{-x}\,dx$$

$$= -\int_{+\infty}^{0} e^{-y}\,dy + \int_{0}^{+\infty} e^{-x}\,dx = 2\int_{0}^{+\infty} e^{-x}\,dx = -2\left(e^{-x}\Big|_{0}^{+\infty} = 2 < \infty\right.$$

1st int.
$y=-x$

Note:

$\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty \iff f \in L_1$,, $\int_{-\infty}^{+\infty} |f(x)|^2\,dx < \infty \iff f \in L_2$

$f \in L_\infty(\mathbb{R}) \implies \max_{\mathbb{R}} |f(x)| < \infty$

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|x|} e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{-i\alpha x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x} e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1-i\alpha)x}\, dx + \int_{0}^{+\infty} e^{-(1+i\alpha)x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1-i\alpha)x}}{1-i\alpha} \right]_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1+i\alpha)x}}{1+i\alpha} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\alpha} - 0 - 0 + \frac{1}{1+i\alpha} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1+i\alpha + 1 - i\alpha}{(1-i\alpha)(1+i\alpha)} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

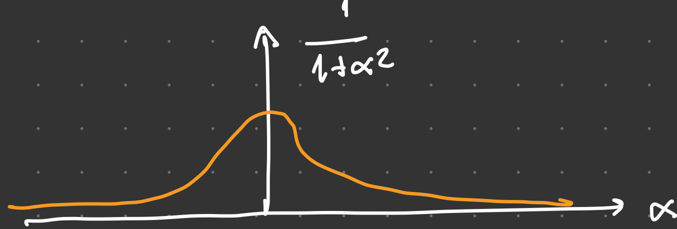
$$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$f'(x) = \begin{cases} -e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0 \end{cases}$$

$f'(x)$ is piecewise-defined

$$\int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha \overset{?}{<} \infty$$

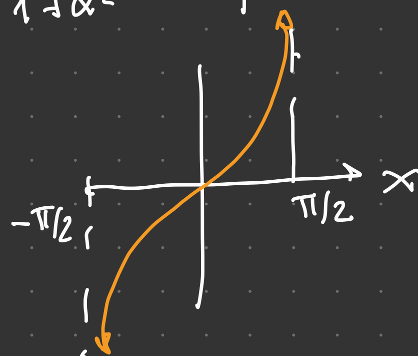
$$\sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \left| \frac{1}{1+\alpha^2} \right| d\alpha = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{d\alpha}{1+\alpha^2} \overset{?}{<} \infty$$



$$\frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}$$

$$\int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} = \Big|\arctan\alpha\Big|_{-\infty}^{+\infty} = \lim_{\alpha\to+\infty}\arctan\alpha - \lim_{\alpha\to-\infty}\arctan\alpha$$

$\tan x$



$$\lim_{x\to\pi/2}\tan x = +\infty$$

$$\lim_{x\to-\pi/2}\tan x = -\infty$$

$$\lim_{x\to\infty}\arctan(x) = \frac{\pi}{2}, \quad \lim_{x\to-\infty}\arctan(x) = -\frac{\pi}{2}.$$

$$\int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} = \pi < \infty$$

Using the inversion theorem, because $f \in C^0(\mathbb{R})$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha$$

$$\downarrow$$

$$f(x) = e^{-|x|} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha \quad \forall x \in \mathbb{R}$$ (because $f \in C^0(\mathbb{R})$)

- $x = 0 \longrightarrow$ $$f(0) = 1 = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+\alpha^2} d\alpha$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} \longrightarrow \int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} = \pi$$

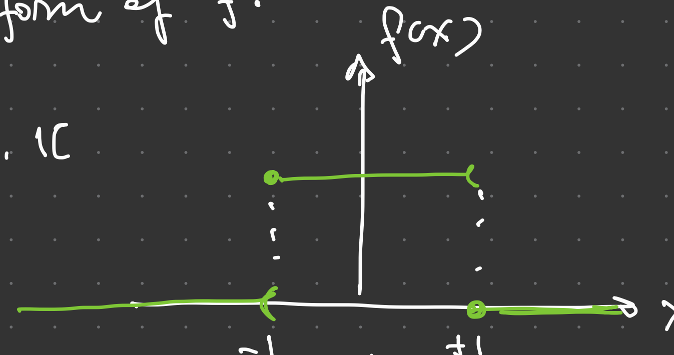
- $x = 1 \longrightarrow$ $$f(1) = e^{-1} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{i\alpha}}{1+\alpha^2} d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\alpha}{1+\alpha^2}\,d\alpha + \frac{i}{\pi}\underbrace{\int_{-\infty}^{\infty}\frac{\sin\alpha}{1+\alpha^2}\,d\alpha}_{=0 \text{ because } \frac{\sin\alpha}{1+\alpha^2} \text{ is odd}}$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\cos\alpha}{1+\alpha^2}\,d\alpha \rightarrow \int_{-\infty}^{+\infty}\frac{\cos\alpha}{1+\alpha^2}\,d\alpha = \frac{\pi}{e}$$

- Example 2: compute the Fourier transform of f:

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1[\\ 0 & \text{otherwise} \end{cases}$$



f and f' are piecewise-defined.

$$f \in L_1(\mathbb{R}) \iff \int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$$

$$\int_{-\infty}^{+\infty} |f(x)|\, dx = \int_{-1}^{+1} 1\, dx = 2 < \infty$$

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1 \cdot e^{-i\alpha x}\, dx$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{e^{-i\alpha x}}{i\alpha} \Big|_{-1}^{+1} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} =$$

$$= \frac{2}{\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{2i\alpha} = \sqrt{\frac{2}{\pi}} \frac{\sin\alpha}{\alpha} = \hat{f}(\alpha)$$

$$e^{+i\alpha} - e^{-i\alpha} = 2i \sin\alpha$$

2.3 Properties of the Fourier transform

$f, g : \mathbb{R} \to \mathbb{R}$ s.t. $\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$ and $\int_{-\infty}^{+\infty} |g(x)|\,dx < \infty$

$\mathcal{F}(f) = \hat{f}$ and $\mathcal{F}(g) = \hat{g}$

2.3.1 Continuity and linearity

- $\hat{f}$ is continuous $\forall\, \alpha \in \mathbb{R}$ and $\lim_{\alpha \to \pm\infty} \hat{f}(\alpha) = 0$.
- $\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$ $\quad \forall\, a, b \in \mathbb{R}$

Proof:

$$\mathcal{F}(af + bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (af + bg)\, e^{-i\alpha x}\, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f e^{-i\alpha x} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g e^{-i\alpha x} dx$$

$$= a \hat{f}(\alpha) + b \hat{g}(\alpha) \quad \square$$

2.8.2 Fourier transform of a convolution product

- Definition: the convolution product of two function f and g is a function denoted $f * g : \mathbb{R} \to \mathbb{R}$ defined as

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-t) g(t) dt$$

It's symmetric $\quad (f*g)(x) = \int_{-\infty}^{+\infty} f(t) g(x-t) dt$

$$\underset{y=x-t}{=} -\int_{+\infty}^{-\infty} f(x-y) g(y) dy = \int_{-\infty}^{\infty} f(x-y) g(y) dy \underset{\substack{\text{rename} \\ y=t}}{=} \int_{-\infty}^{+\infty} f(x-t) g(t) dt$$

$$\mathcal{F}(f*g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) = \sqrt{2\pi}\,\hat{f}\,\hat{g}$$

Proof:

$$\mathcal{F}(f*g)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (f*g)(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x-t)g(t)\,dt \right] e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x-t)\, e^{-i\alpha x}\, dx \right) g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha(y+t)}\, dy \right) g(t)\, dt$$

$y := x - t$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\alpha y} dy}_{\hat{f}(\alpha)} \underbrace{\int_{-\infty}^{+\infty} g(t) e^{-i\alpha t} dt}_{\sqrt{2\pi}\, \hat{g}(\alpha)}$$

$$= \sqrt{2\pi}\, \hat{f}(\alpha)\, \hat{g}(\alpha) \quad \square$$

2.3.3 Fourier transform of derivative

- $f \in C^1(\mathbb{R})$ (i.e., $f' \in C^0(\mathbb{R})$) and $\int_{-\infty}^{+\infty} |f'(x)| \, dx < \infty$

then: $\mathcal{F}(f')(\alpha) = i\alpha\, \mathcal{F}(f)(\alpha)$

If $f \in C^n(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |f^{(k)}(x)|\,dx < \infty$ for $k = 0, 1, \dots, n$

then $\mathcal{F}(f^{(k)})(\alpha) = (i\alpha)^k \mathcal{F}(f)(\alpha)$ $\forall\, \alpha \in \mathbb{R}$ and $k = 1, \dots, n$.

$$\widehat{f^{(k)}}(\alpha) = (i\alpha)^k \hat{f}(\alpha)$$

Proof:

$$\mathcal{F}(f')(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x)\, e^{-i\alpha x}\, dx =$$

$u = e^{-i\alpha x}$

$v = f$

$du = -i\alpha\, e^{-i\alpha x}\, dx$

$dv = f'\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left(f(x) e^{-i\alpha x} \right)\Big|_{-\infty}^{+\infty} + i\alpha \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx. = (*)$$

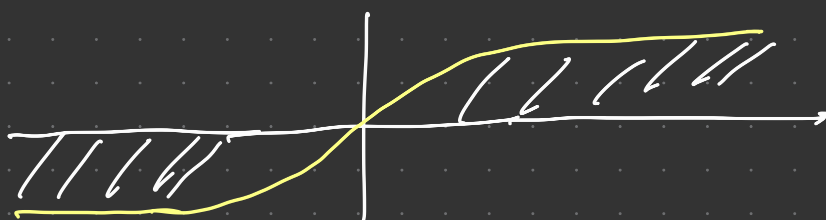
$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\lim_{x \to \pm\infty} |f(x) e^{-i\alpha x}| = \lim_{x \to \pm\infty} |f(x)| = 0$$

$$|e^{-i\alpha x}| = 1 \quad \forall x \in \mathbb{R}$$

because

$$\int_{-\infty}^{+\infty} |f(x)| dx < \infty$$



$$(*) = i\alpha \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx = i\alpha \hat{f}(\alpha)$$