$$\int_\Gamma F \cdot dl \neq 0 \rightarrow$$ F does not derive from a potential in Ω_4

01/10/2024

$$\left.\begin{array}{l}
\displaystyle\lim_{\substack{y \to 0^+ \\ x<0}} f_3(x,y) = \lim_{\substack{y \to 0^+ \\ x<0}} -\operatorname{arctg}\left(\frac{x}{y}\right) + \pi + c_2 \\[2ex]
\qquad \| \\[2ex]
\displaystyle\lim_{\substack{y \to 0^- \\ x<0}} f_3(x,y) = \lim_{\substack{y \to 0^- \\ x<0}} -\operatorname{arctg}\left(\frac{x}{y}\right) + c_2
\end{array}\right\} \rightarrow \pi + c_2 \neq c_2$$

2.4 Green's theorem

All the results of this section are for $\mathbb{R}^2$ $(n=2)$.

2.4.1 Recalls, notation, and definitions

- $D_r(x) = \{ y \in \mathbb{R}^2 : \|x-y\|^2 < r^2 \}$

(an open disk in $\mathbb{R}^2$, centered at x, with radius r)

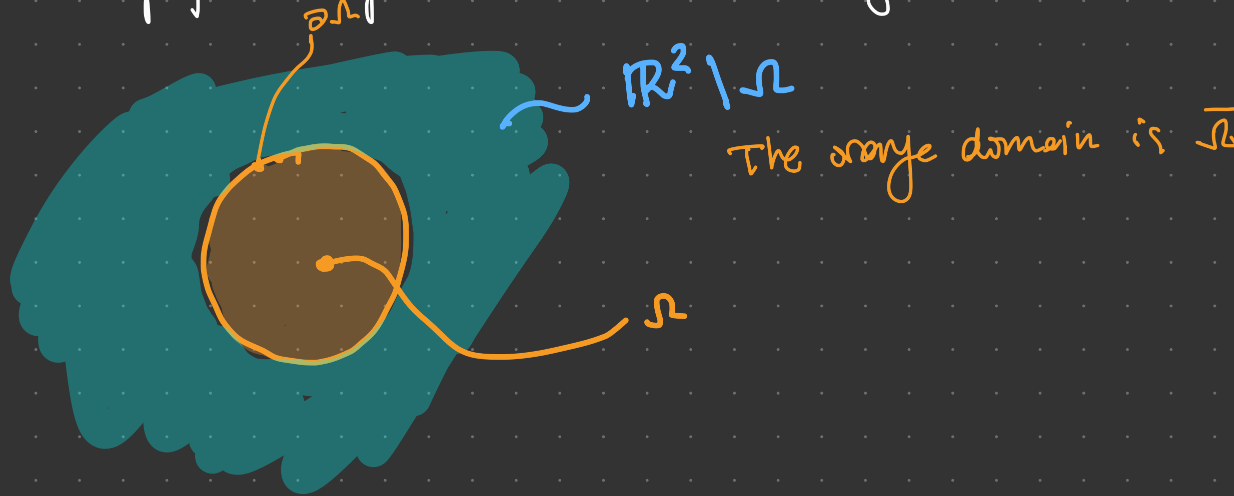
Ω is an open bounded domain in $\mathbb{R}^2$.

- We denote by $\partial\Omega$ the boundary of Ω

$\partial\Omega = \{ x \in \mathbb{R}^2 : \Omega \cap D_r(x) \neq \emptyset$ and

$(\mathbb{R}^2 \setminus \Omega) \cap D_r(x) \neq \emptyset, \forall r > 0 \}$

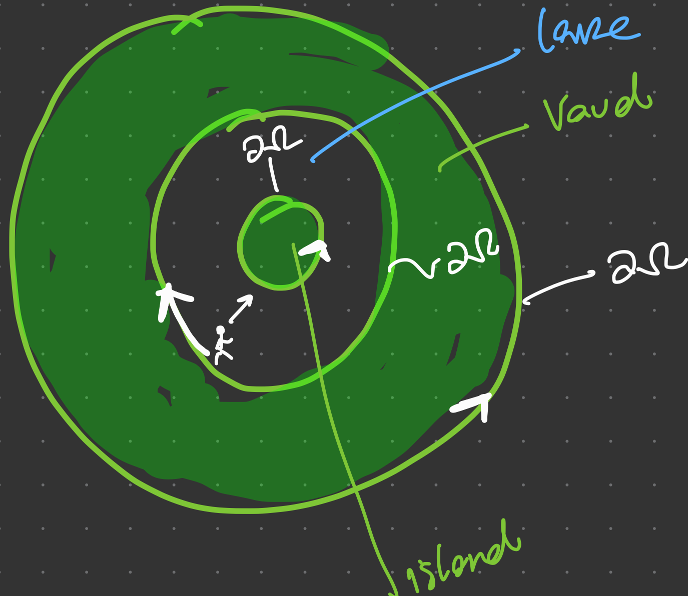
- We denote $\bar{\Omega} = \Omega \cup \partial\Omega$ as the <u>closure of Ω</u> :
the set of points of $\mathbb{R}^2$ that do not belong to the interior of $\mathbb{R}^2 \setminus \Omega$

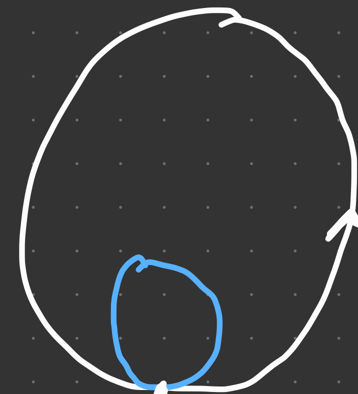


Definitions

1) Let $\Omega \subset \mathbb{R}^2$ a bounded an open s.t. $\partial\Omega$ is a closed (piecewise) simple regular curve. We say that $\partial\Omega$ is <u>positively (or negatively) oriented</u> if when we tour along

the curve the domain is on the left (respect., the right)

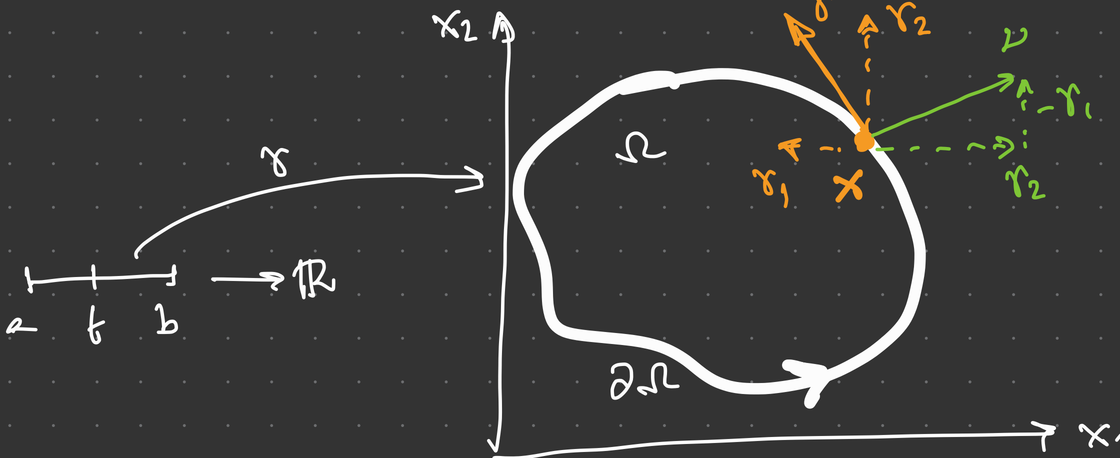


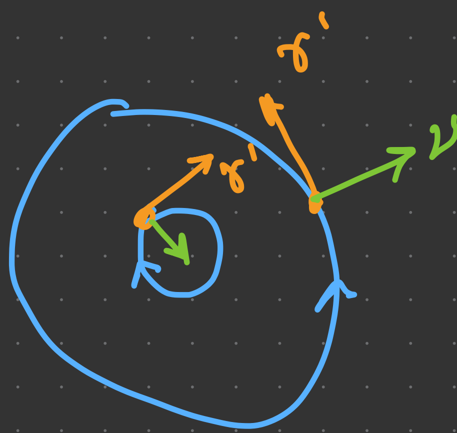


Meaning of the positive (or negative) orientation:

For a parameterization $\gamma : [a,b] \to \partial\Omega$
$$t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$$
of $\partial\Omega$, the normal vector of $\partial\Omega$ at x given by

$\nu(x) = (\gamma_2'(t), -\gamma_1'(t))$ is an external normal of Ω



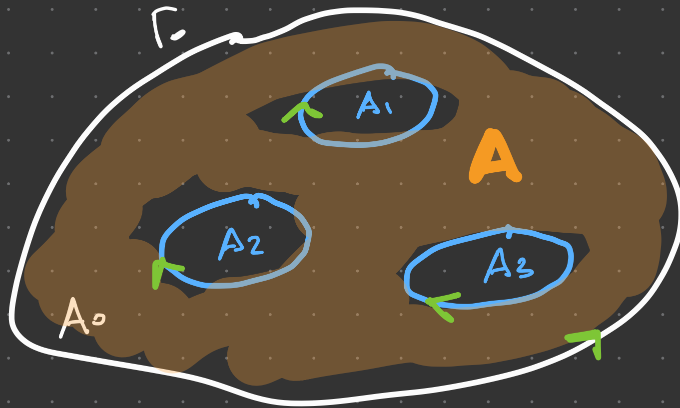


2) We say that a bounded open domain $A \subset \mathbb{R}^2$ is a <u>regular domain</u> if $\exists$ bounded open domains.

$A_0, A_1, A_2, \dots, A_m \subset \mathbb{R}^2$ s.t.

- $\bar{A}_j \subset A_0 \quad , \forall j = 1, 2, \dots, m$
- $\bar{A}_i \cap \bar{A}_j = \emptyset \quad \forall i, j = 1, \dots, m$, and $i \neq j$
- $A = A_0 \setminus \bigcup_{j=1}^{m} \bar{A}_j$
- $\partial A = \cup \Gamma_j$, $\Gamma_j = \partial A_j$, $j = 0, \dots, m$ and Γ_j are

(piecewise) simple regular curves.



2.4.2 Green's theorem

Theorem:

let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $F: \bar{A} \to \mathbb{R}^2$

$(x,y) \longmapsto F(x,y) = (F_1(x,y), F_2(x,y))$

be a vector field s.t. $F \in C^1(\bar{A}, \mathbb{R}^2)$. Then

$$\iint_A \operatorname{curl} F(x,y)\,dx\,dy = \iint_A \left(\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right) dx\,dy$$

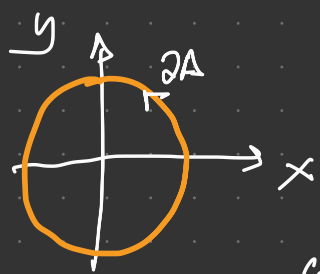
$$= \int_{\partial A} F \cdot dl$$

Remark:

- If F derives from a potential $\Rightarrow \operatorname{curl} F = 0 \Rightarrow \int_{\partial A} F \cdot dl = 0$ (same results as in Theorem 2 of potential fields).

2.4.3 Examples:

Example 1: Verify Green's theorem for $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ and $F(x,y) = (y^2, x)$



∂A positively oriented

1) $\iint_A \operatorname{curl} F(x,y)\,dx\,dy$

$$\operatorname{curl} F(x,y) = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y^2) = 1-2y$$

$$\iint_A (1-2y)\,dx\,dy = \int_0^1 \int_0^{2\pi} (1-2r\sin\theta)\, r\,d\theta\,dr$$

$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

$$= \int_0^1 \left(\int_0^{2\pi} r\,d\theta - \int_0^{2\pi} 2r^2 \sin\theta\,d\theta \right) dr$$

$$\left(\int_0^{2\pi} \sin\theta\,d\theta = 0 \right)$$

$$= \int_0^1 r \, 2\pi = \pi r^2 \Big|_0^1 = \pi$$

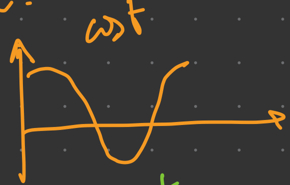
2) $\displaystyle\int_{\partial A} F \cdot dl$, $\quad \gamma(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$

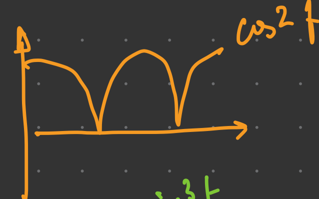
$\quad \gamma'(t) = (-\sin t, \cos t)$

$$\int_{\partial A} F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} (\sin^2 t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

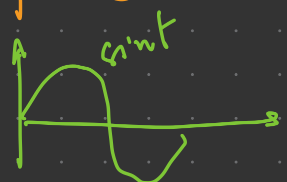
$$= -\int_0^{2\pi} \sin^3 t \, dt + \int_0^{2\pi} \cos^2 t \, dt.$$

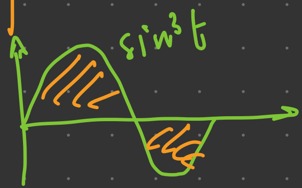
Note:





$$\int_0^{2\pi} \cos^2 t \neq 0.$$





$$\int_0^{2\pi} \sin^3 t = 0.$$

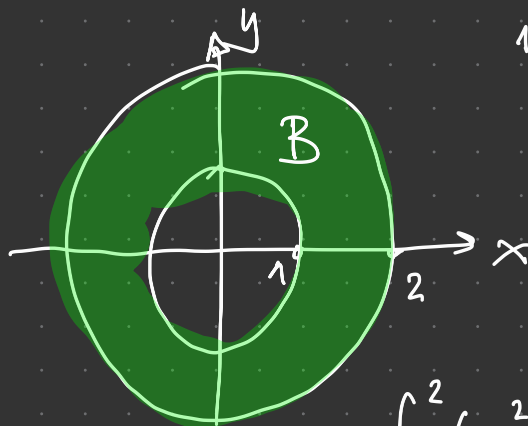
- $\displaystyle\int_0^{2\pi} \sin^3 t \, dt = \int_0^{2\pi} \sin t \, \underbrace{(1 - \cos^2 t)}_{\sin^2 t} \, dt = \underbrace{\cancel{\int_0^{2\pi} \sin t}}_{0} - \int_0^{2\pi} \sin t \cos^2 t \, dt$

$$= \frac{1}{3} \cos^3 t \Big|_0^{2\pi} = 0$$

- $\displaystyle\int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t) \, dt = \int_0^{2\pi} \frac{1}{2} \, dt + \int_0^{2\pi} \frac{1}{2} \cos 2t \, dt$

$$= \pi + 0 = \pi \; //$$

Example 2: Verify Green's theorem for $B = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4\}$ and $F(x,y) = (x^2 y, 2xy)$



1) $\iint_B \operatorname{curl} F(x,y)\, dx\, dy$

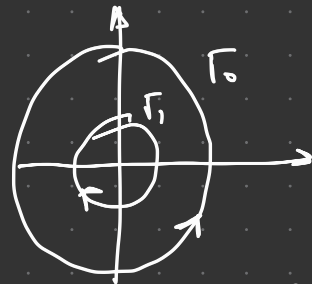
$\operatorname{curl} F(x,y) = 2y - x^2$

$$\iint_B 2y - x^2 \, dx\, dy = \int_1^2 \int_0^{2\pi} (2r\sin\theta - r^2\cos^2\theta) \, r\, dr\, d\theta$$

$$= \int_1^2 \Big(2r^2 \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{0} - r^3 \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{\pi} \Big) dr$$

$$= -\pi \int_1^2 r^3 dr = -\frac{\pi}{4} r^4 \Big|_1^2 = \frac{-15\pi}{4}$$

2)



$\partial B = \Gamma_0 \cup \Gamma_1$

$\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=4\}$

$\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$

$\gamma_0(t) = (2\cos t, 2\sin t), \quad t \in [0, 2\pi]$

$\gamma_1(t) = (\cos t, -\sin t), \quad t \in [0, 2\pi]$

$\gamma_0'(t) = 2(-\sin t, \cos t)$

$\gamma_1'(t) = -(\sin t, \cos t)$

$$\int_{\partial B} F \cdot dl = \int_{\Gamma_0} F \cdot dl + \int_{\Gamma_1} F \cdot dl$$

$$\int_{\Gamma_0} F \cdot dl = \int_0^{2\pi} F(\gamma_0(t)) \cdot \gamma_0'(t)\, dt = \int_0^{2\pi} (8\cos^2 t \sin t, 8\cos t \sin t) \cdot (-2\sin t, 2\cos t)\, dt$$

$$= 16\int_0^{2\pi} \left(-\cos^2 t \sin^2 t + \underbrace{\cos^2 t \sin t}_{=0} \right) dt$$

$$\int_0^{2\pi} \cos^2 t \sin^2 t \, dt \int_0^{2\pi} \left(\tfrac{1}{2}\sin 2t\right)^2 dt = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos 4t}{2}\, dt = \frac{\pi}{4}$$

$$\int_{\Gamma_0} F \cdot dl = -4\pi$$

$$\int_{\Gamma_1} F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt = \; ---- \; = \underbrace{\int_0^{2\pi} \cos^2 t \sin^2 t \, dt}_{\pi/4} + 2\underbrace{\int_0^{2\pi} \cos^2 t \sin t \, dt}_{=0}$$

$$= \pi/4$$

Then $\int_{\partial B} F \cdot dl = -4\pi + \frac{\pi}{4} = \frac{-15\pi}{4}$ //