17/12/2024

Question 1 The non-zero complex Fourier coefficients of the function $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x) = \cos(x) + 3\sin(3x),$$

which enables to express g as:

$$g(x) = \sum_{k=-\infty}^{\infty} = c_k e^{ikx}$$

are

- [x] c_1, c_{-1}, c_3, c_{-3}
- [ ] c_1, c_{-3}
- [ ] c_1, c_3
- [ ] c_0, c_1, c_{-1}, c_3, c_{-3}

$$Fg(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$T = 2\pi$

$$C_n = \frac{a_n - ib_n}{2}, \quad C_{-n} = \frac{a_n + ib_n}{2}$$

$a_0 = 0$, $a_1 = 1$, $a_n = 0 \quad \forall\, n > 1$

$b_1 = b_2 = 0$, $b_3 = 3$, $b_n = 0 \quad \forall\, n > 3$

$c_1 = \frac{a_1}{2} = 1/2 \neq 0$, $\quad c_{-1} = \frac{a_1}{2} = 1/2 \neq 0$

$c_3 = \frac{-i3}{2} \neq 0$, $\quad c_{-3} = \frac{+i3}{2} \neq 0$

Question 2 Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (x^2 + y^2 + z^2, xy, z).$$

Then:

- [x] $\text{div}(\text{curl}(F)) = 0$ over $\mathbb{R}^3$
- [ ] $\text{div}(F) = 0$ over $\mathbb{R}^3$
- [ ] $F = \nabla f$, for any $f \in C^1(\mathbb{R}^3)$
- [ ] $\text{curl}(F) = 0$ over $\mathbb{R}^3$

$$\text{div } F = 2x + x + 1 \neq 0$$

$$\text{curl } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= (0 - 0, 2z - 0, y - 2y) = (0, 2z, -y) \neq 0$$

$$\text{div curl } F = 0 + 0 + 0 = 0$$

$\square$ $\mathrm{curl}(F) = 0$ over $\mathbb{R}$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto (x, y),$$

$F \in C^1(\mathbb{R}^2)$

and let $R \in \mathbb{R}$, $R > 0$ and A be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of A by ∂A, and the outer unit normal of ∂A by $\nu : \partial A \to \mathbb{R}^2$.

The integral $\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- [ ] 0
- [x] $2\pi R^2$
- [ ] $\frac{3}{4}\pi R$
- [ ] $\frac{3}{5}\pi^2 R$

$$\int_A \mathrm{div}\, F \, dA = \int_{\partial A} F \cdot \nu \, dl \quad , \quad \mathrm{div}\, F = 1 + 1 = 2$$

$$\int_{\partial A} F \cdot \nu \, dl = \int_A 2 \, dA = 2 \,\mathrm{area}(A) = 2\pi R^2$$

Question 4 Let F be the vector field defined by:

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

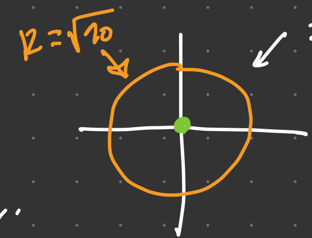
F is conservative (*i.e.* it derives from a potential)

- ☒ over $\Omega = \{(x, y) : 1 \leq x \leq 3, 2 \leq y \leq 10\}$.
- ☐ over $\Omega = \{(x, y) : 2 \leq x^2 + y^2 \leq 4\}$. ?
- ☐ over $\Omega = \{(x, y) : x^2 + y^2 \leq 10\}$.
- ☐ for any domain Ω.

curl $F = 0$? , $\text{curl } F(x,y) = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$

$$\text{curl } F(x,y) = \frac{1}{(x^2+y^2)^2}\left(x^2+y^2-2x^2+x^2+y^2-2y^2\right) = 0$$

F is not defined at •



- Necessary condition ✓
- Ω is convex and/or simply connected.

F is not defined at $(0,0)$

F is defined in Ω

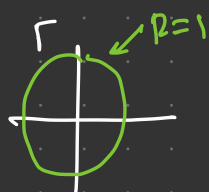
it is not simply-connected, it is not convex.

F is defined in Ω

Ω is convex and is simply-connected.

Note: to show that F does not derive from a potential

We need to find a closed curve $\Gamma \subset \Omega$ s.t. $\int_\Gamma F \cdot dl \neq 0$.



$$\gamma(\theta) = (\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi]$$

$$\gamma'(\theta) = (-\sin\theta, \cos\theta)$$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(\theta)) \cdot \gamma'(\theta)\, d\theta$$

$$F(\gamma(\theta)) = \frac{1}{\cos^2\theta + \sin^2\theta}\left(-\sin\theta,\ \cos\theta\right) = (-\sin\theta,\ \cos\theta)$$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta)\, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi \neq 0.$$

Question 5 Let $T > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{T}{2}\right[\\ -1 & \text{if } x \in \left[\frac{T}{2}, T\right[\end{cases}$$

extended by T-periodicity to $\mathbb{R}$. Its Fourier series is:

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

The sum $\sum_{n=0}^{\infty} \frac{1}{n^2+n+1/4}$ is equal to:

- [ ] 0
- [x] $\pi^2/2$
- [ ] $\pi^2/16$
- [ ] $\pi^2/8$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+n+1/4} = \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2}$$

Parceval :

$$\frac{2}{T}\int_0^T f(x)^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

$a_0 = 0$, $a_n = 0 \quad \forall n > 0.$,

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{2\pi}{T}(2n+1)x\right) = \sum_{m \text{ is odd}} \frac{4}{\pi m} \sin\left(\frac{2\pi}{T} m x\right)$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1/2} \sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

$$\int_0^T f(x)^2 dx = \int_0^{T/2} (1)^2 dx + \int_{T/2}^{T} (-1)^2 dx = \int_0^T 1 \, dx = T$$

$$\frac{2}{T} \int_0^T f(x)^2 dx = 2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=0}^{\infty} \frac{4}{\pi^2} \frac{1}{(n+1/2)^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2} = \frac{1}{2} \pi^2.$$

Question 6 Let $u : \mathbb{R} \to \mathbb{R}$ be a C^1-periodic solution of the following system:

$$\begin{cases} u'(x) + 3u(x) = \cos(3x) + \sin(5x), & \forall x \in \mathbb{R} \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases}$$

Then:

- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{6}\cos(3x) - \frac{5}{34}\sin(5x) + \frac{3}{28}\cos(5x)$
- [x] $u(x) = \frac{1}{6}\sin(3x) + \frac{1}{6}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{5}{34}\cos(5x)$
- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{8}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{3}{28}\cos(5x)$
- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{3}{28}\cos(5x)$

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$u'(x) = \sum_{n=1}^{\infty} [-n a_n \sin nx + n b_n \cos nx]$$

$$u'(x) + 3u(x) = \cos(3x) + \sin(5x)$$

$$\frac{3}{2} a_0 + (n b_n + 3 a_n) \cos nx + (3 b_n - n a_n) \sin nx = \cos 3x + \sin 5x$$

- $a_0 = 0$
- $n=3:$ $\quad 3 b_3 + 3 a_3 = 1 \quad , \quad 3 b_3 - 3 a_3 = 0 \longrightarrow a_3 = b_3 = 1/6$
- $n=5$ $\quad 5 b_5 + 3 a_5 = 0 \quad , \quad 3 b_5 - 5 a_5 = 1 \longrightarrow a_5 = -\frac{5}{34} \, , \, b_5 = \frac{3}{34}$

Question 7 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\int_\Gamma f\, dl$ is equal to:

- [ ] 0
- [x] $2\pi R$
- [ ] $2\pi R^2 + \pi R + 1$
- [ ] $2\pi R^3 + \pi R^2 + R$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$\theta \longmapsto (R\cos\theta, R\sin\theta)$

$\gamma'(\theta) = R(-\sin\theta, \cos\theta)$

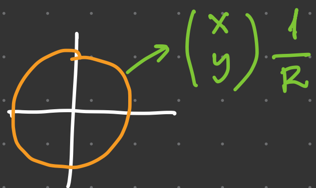
$\|\gamma'(\theta)\| = R$

$\int_\Gamma f \cdot dl = \int_0^{2\pi} f(\gamma(\theta))\, \|\gamma'(\theta)\|\, d\theta = \int_0^{2\pi} (R^2 \cos\theta \sin\theta + R\cos\theta + 1)\, R\, d\theta$

$$= \int_0^{2\pi} R^3 \cos\theta \sin\theta \, d\theta + \underbrace{\int_0^{2\pi} R^2 \cos\theta \, d\theta}_{=0} + \int_0^{2\pi} R \, d\theta$$

$$= \underbrace{R^3 \frac{1}{2} \sin^2\theta \Big|_0^{2\pi}}_{=0} + 0 + 2\pi R = 2\pi R$$

Note: $f(x,y) = xy + x + 1 = F(x,y) \cdot \nu(x,y)$



$$F(x,y) \cdot \nu(x,y) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \frac{1}{R}$$

F_1, F_2

$$F_1 = 1 + \frac{1}{x}$$

$$F_2 = x$$

$$\int_\Gamma f \, dl = \int_{\Gamma = \partial A} F \cdot \nu \, dl = \int_A \operatorname{div} F$$

Question 8 Consider the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ defined such that:

$$g(x) = e^{-\frac{x^2}{2}}, \quad \text{and } h(x) = g\left(\frac{x}{4}\right).$$

Then, the Fourier transform $\hat{h} = \mathcal{F}(h)$ verifies:

- ☐ $\hat{h}'(\alpha) = i\frac{\alpha}{4}e^{-\frac{\alpha^2}{32}}$
- ☐ $\int_{-\infty}^{\infty} |\hat{h}'(\alpha)| \mathrm{d}\alpha = \sqrt{2\pi}$
- ☐ $\hat{h}(\alpha) = e^{-8\alpha^2}$
- ☒ $\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[e^{4\alpha^2}\hat{h}(\alpha)\right]\right)^2 = 64\alpha^2\hat{h}(\alpha)$

$h(x) = e^{-x^2/32}$, $\omega = \frac{1}{\sqrt{32}}$

8	$f(y) = e^{-w^2y^2}$	$(w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w \rvert} e^{-\frac{\alpha^2}{4w^2}}$

$\hat{h}(\alpha) = \frac{\sqrt{32}}{\sqrt{2}} e^{-\frac{32}{4}\alpha^2} = 4e^{-8\alpha^2}$

$$\hat{h}'(\alpha) = -64\,\alpha\, e^{-8\alpha^2}$$

$$\int_{-\infty}^{\infty} |\hat{h}'(\alpha)|\, d\alpha = \int_{-\infty}^{0} |\hat{h}'(\alpha)|\, d\alpha + \int_{0}^{+\infty} |\hat{h}'(\alpha)|\, d\alpha$$

$$= 64 \int_{-\infty}^{0} -\alpha\, e^{-8\alpha^2}\, d\alpha + |-64| \int_{0}^{\infty} \alpha\, e^{-8\alpha^2}\, d\alpha$$

$$= 128 \int_{0}^{\infty} \alpha\, e^{-8\alpha^2}\, d\alpha = 128\, \frac{-1}{16}\, e^{-8\alpha^2} \Big|_{0}^{\infty} = \frac{128}{16}.$$

- $$\left(\frac{1}{2} \frac{d}{d\alpha} \left(e^{4\alpha^2} \hat{h}(\alpha) \right) \right)^2 = 64\, \alpha^2\, \hat{h}(\alpha)$$

$$e^{4\alpha^2} \hat{h}(\alpha) = 4 e^{-4\alpha^2}, \qquad \frac{d}{d\alpha}\left(4 e^{-4\alpha^2}\right) = -32\,\alpha\, e^{-4\alpha^2}$$

$$\left(-16\alpha\, e^{-4\alpha^2}\right)^2 = 16^2 \alpha^2 e^{-8\alpha^2} = 4 e^{-8\alpha^2} 64\alpha^2 = 64\alpha^2 \hat{h}(\alpha) \checkmark$$

Question 14: *This question is worth 8 points.*

$\square_0$ $\square_1$ $\square_2$ $\square_3$ $\square_4$ $\square_5$ $\square_6$ $\square_7$ $\square_8$

(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \to \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} \Big(9u(t) - 4u''(t)\Big) e^{-\frac{3}{2}|x-t|} \mathrm{d}t = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

$$-10u(x) + \left(9u(x) - 4u''(x)\right) * \underbrace{e^{-\frac{3}{2}|x|}}_{h(x)} = \underbrace{\frac{4x^2}{(2\pi + x^2)^2}}_{g(x)}$$

$$\mathcal{F}\left(-10u(x) + \left(9u(x) - 4u''(x)\right) * h(x)\right)(\alpha)$$
$$= \mathcal{F}(g(x))(\alpha)$$

$$-10\hat{u}(\alpha) + \sqrt{2\pi}\,\mathcal{F}\left(9u(x) - 4u''(x)\right)(\alpha)\,\hat{h}(\alpha) = \hat{g}(\alpha)$$

$$-10\hat{u}(\alpha) + \sqrt{2\pi}\left(9\hat{u}(\alpha) - 4(i\alpha)^2\hat{u}(\alpha)\right)\hat{h}(\alpha) = \hat{g}(\alpha)$$

$$\hat{h}(\alpha) = \frac{3}{2}\sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha^2 + 9/4}$$

$$-10\hat{u}(\alpha) \overbrace{+\sqrt{2\pi}\left(9\hat{u}(\alpha) + 4\alpha^2\hat{u}(\alpha)\right)\cdot\frac{3}{2}\sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha^2 + 9/4}}^{12} = \hat{g}(\alpha)$$

$$\hat{u}(\alpha)(12 - 10) = \hat{g}(\alpha) \Rightarrow \hat{u}(\alpha) = \frac{1}{2}\hat{g}(\alpha)$$

Inverse Fourier: $u(x) = \frac{1}{2}g(x) = \dfrac{2x^2}{(2\pi + x^2)^2}.$