Then $\lambda = \left(\frac{n\pi}{L}\right)^2$.

The sturm-Liouville problem only has non-trivial solutions for $\lambda = \left(\frac{n\pi}{L}\right)^2$ and the solution are:

$$u(t) = u_n \sin\left(\frac{n\pi}{L} t\right)$$ where $u_n \in \mathbb{R}$ is an arbitrary constant

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Example: same problem, but with different boundary conditions

$$y''(t) + \lambda y(t) = 0 \quad \forall t \in]0, L[$$
$$y'(0) = y'(L) = 0$$

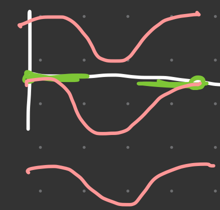
- Case 1 $\lambda = 0$ $\quad y''(t) = 0 \quad \forall t \;]0, L[$
  $+ BC$

Possible solution $y(t) = y_1 t + y_0$, $y_1, y_0 \in \mathbb{R}$.

$y'(t) = y_1$ to fulfill BCs then

$$y'(0) = y'(L) = y_1 = 0 \longrightarrow y_1 = 0.$$

$$y(t) = 0 \cdot t + y_0 \longrightarrow y(t) = y_0$$



- Case 2 : $\lambda < 0$

The solution is like $y(t) = y_0 \cosh(t\sqrt{-\lambda})$

$$y'(t) = y_0 \sqrt{-\lambda} \sinh(t\sqrt{-\lambda}), \quad y''(t) = -y_0 \lambda \cosh(t\sqrt{-\lambda})$$

$$y''(t) + \lambda y(t) = (\underbrace{-y_0\lambda + \lambda y_0}_{=0}) \cosh(t\sqrt{-\lambda}) = 0$$

What about BCs?

$$y'(0) = y_0\sqrt{-\lambda}\, \underbrace{\sinh(0)}_{=0} = 0 \ \checkmark$$

$$y'(L) = y_0\sqrt{-\lambda}\, \underbrace{\sinh(L\sqrt{-\lambda})}_{\neq 0\ (\lambda \neq 0)} = 0 \implies \underset{(\lambda \neq 0)}{y_0\sqrt{-\lambda}} = 0 \implies y_0 = 0$$

the trivial solution!

Case 3: $\lambda > 0$

$y(t) = y_0 \cos(t\sqrt{\lambda})$, $y'(t) = -y_0\sqrt{\lambda}\sin(t\sqrt{\lambda})$

$y''(t) = -y_0 \lambda \cos(t\sqrt{\lambda})$

$y''(t) + \lambda y(t) = -y_0 \lambda \cos(t\sqrt{\lambda}) + \lambda y_0 \cos(t\sqrt{\lambda}) = 0$ ✓

What about BCs?

$y'(0) = -y_0\sqrt{\lambda}\,\underbrace{\sin(0\sqrt{\lambda})}_{=0} = 0$ ✓

$y'(L) = -y_0\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0 \Rightarrow \sin(L\sqrt{\lambda}) = 0$

$\Rightarrow L\sqrt{\lambda} = n\pi \quad n \in \mathbb{N} \longrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

and the associated solution is $y_n(t) = \beta_n \cos\left(\frac{n\pi}{L} t\right)$

where $\beta_n \in \mathbb{R}$ is an arbitrary constant.

3.2.3 Application of Fourier series

Diff. equations with additional data of the form

$u(0) = u(T)$, $u(x) = u(x+T)$, i.e., u is T-periodic.

Heat diffusion:

$$\frac{\partial^2 u}{\partial \theta^2} = R^2 f(R\cos\theta, R\sin\theta$$

$$u(0) = u(2\pi)$$

$$u'(0) = u'(2\pi)$$

u: Temperature.

Example 1:

let $f: \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and 2π-periodic

let $m, k \in \mathbb{R}$ s.t. $m \neq 0$. Find a solution $y = y(t)$

of the problem:

$$m y''(t) + k y(t) = f(t) \quad \forall\ t \in]0, 2\pi[$$

$$y(0) = y(2\pi) \text{ and } y'(0) = y'(2\pi)$$

Particular case: $f(t) = \cos(t)$.

look for periodic solutions:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nt) + b_n \sin(nt) \right]$$

Because f is 2π-periodic:

$$Ff(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos nt + \beta_n \sin nt \right]$$

α_n and β_n are known (f is given, so I can compute them).

$$u'(t) = \sum_{n=1}^{\infty} \left[-a_n\, n \sin wt + b_n\, n \cos nt \right]$$

$$u''(t) = -\sum_{n=1}^{\infty} n^2 \left(a_n \cos nt + b_n \sin nt \right)$$

$$m y''(t) + k y(t) = f(t) = F f(t)$$

$$\frac{k a_0}{2} + \sum_{n=1}^{\infty} \left[(k - m n^2) a_n \cos nt + (k - m n^2) b_n \sin nt \right]$$

$$= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos nt + \beta_n \sin nt \right]$$

$$\frac{k a_0}{2} = \frac{\alpha_0}{2}, \quad (k - m n^2) a_n = \alpha_n, \quad (k - m n^2) b_n = \beta_n$$

$$a_0 = \frac{\alpha_0}{k}, \quad a_n = \frac{\alpha_n}{k - m n^2}, \quad b_n = \frac{\beta_n}{k - m n^2}$$

For the case $f(t) = \cos t \longrightarrow \alpha_0 = ?\ \alpha_n = ?\ \beta_n = ?$

$$f(t) = \cos t = Ff(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos nt + \beta_n \sin nt \right]$$

$$\alpha_0 = 0, \quad \beta_n = 0, \quad \alpha_1 = 1, \quad \alpha_n = 0 \quad \forall\, n > 1.$$

$$a_0 = 0, \quad a_1 = \frac{1}{K-m}, \quad a_n = 0 \ \forall\, n > 1 \quad b_n = 0 \ \forall\, n \in \mathbb{N}$$

$$\boxed{u(t) = \frac{1}{K-m} \cos t}$$

Example 2:

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$f(x) = x^2 - \frac{1}{12}$ if $x \in [-1/2, 1/2[$ extended by 1-periodicity

Find $u = u(t)$ solution of

$$u'(t) = f(t) \quad \forall\, x \in]-1/2, 1/2[$$

$$u(-1/2) = u(1/2)$$

Example 3:

let be $\alpha \neq \pm 1$ and $f: \mathbb{R} \to \mathbb{R}$ is 2π-periodic

and piecewise defined in $[0, 2\pi[$. Find $u: [0, 2\pi] \to \mathbb{R}$. !

$$u(t) + \alpha u(t - \pi) = f(t) \quad \forall t \in]0, 2\pi[$$

$$u(0) = u(2\pi). \quad \text{s.t. } u \text{ is } 2\pi\text{-periodic.}$$

3.2.4 Applications of Fourier transform

Example: Find $u(x)$ s.t:

$$u''(x) - u(x) = \underbrace{e^{-|x|}}_{f(x)}$$

$$\mathcal{F}(u''(x) - u(x))(\alpha) = \mathcal{F}(e^{-|x|})(\alpha)$$

linear.

$$\mathcal{F}(u''(x))(\alpha) - \mathcal{F}(u(x))(\alpha) = \mathcal{F}(e^{-|x|})(\alpha)$$

$($

derivat. ↓

$$(i\alpha)^2\, \mathcal{F}(u(x))(\alpha) - \mathcal{F}(u(x))(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{1}{1+\alpha^2}$$

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert \alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = y e^{-w^2 y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

$w = 1$

$$\hat{u}(\alpha)\,(-\alpha^2-1) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha^2+1}$$

$$\hat{u}(\alpha) = -\sqrt{\frac{2}{\pi}}\,\frac{1}{(\alpha^2+1)^2}$$

(Note: to compute inverse Fourier transf.

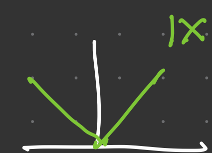
$$\int_{-\infty}^{+\infty}\left|-\sqrt{\frac{2}{\pi}}\,\frac{1}{(1+\alpha^2)^2}\right| d\alpha < \infty$$)

Using Wolfram Alpha

$$\mathcal{F}^{-1}\left(-\sqrt{\frac{2}{\pi}}\,\frac{1}{(1+\alpha^2)^2}\right) = -\frac{1}{2}e^{-|x|}(1+|x|)$$

$$u(x) = -\frac{1}{2}e^{-|x|}(1+|x|).$$

If $x \neq 0$, $\frac{d}{dx}|x| = \mathrm{sign}(x)$, $\frac{d^2}{dx^2}|x| = 0.$



$$u'(x) = \frac{1}{2}\,\mathrm{sign}(x)\, e^{-|x|}(1+|x|) - \frac{1}{2}\,\mathrm{sign}(x)\, e^{-|x|}$$

$$= \frac{1}{2}\,\mathrm{sign}(x)\, e^{-|x|}\,|x|$$

$$u''(x) = \frac{1}{2}\underbrace{(\mathrm{sign}(x))^2}_{=1} e^{-|x|} - \frac{1}{2}\underbrace{(\mathrm{sign}(x))^2}_{=1} e^{-|x|}\,|x|$$

$$= \frac{1}{2} e^{-|x|}(1-|x|).$$

$$u''(x) - u(x) = \frac{1}{2}(1-|x|)\, e^{-|x|} + \frac{1}{2}(1+|x|)\, e^{-|x|} = e^{-|x|} \checkmark$$

Example 2: Find $u(x)$. s.t.

$$9\,u(x) + \underbrace{\int_{-\infty}^{+\infty} 8\,u(t)\, e^{-|x-t|}\,dt}_{\text{convolution.}} = e^{-|x|}$$

3.2.5 Incompatibility of Fourier series and Fourier transform methods

If we can apply both methods:

- u is piecewise-defined
- u is T-periodic
- $\int_{-\infty}^{+\infty} |u(x)|\,dx < \infty$

Then
$$\int_{-\infty}^{+\infty} |u(x)|\,dx = \sum_{n\in\mathbb{Z}} \int_{nT}^{(n+1)T} |u(x)|\,dx \overset{\substack{y = x - nT \\ dy = dx}}{=} \sum_{n\in\mathbb{Z}} \int_{0}^{T} |u(y+nT)|\,dy$$

$u(z+T) = u(z)$

$$= \sum_{n\in\mathbb{Z}} \overbrace{\int_0^T |u(y)|\,dy}^{I}$$

Then $\int_{-\infty}^{+\infty} |u(x)|\, dx = \sum_{n \in \mathbb{Z}} I < \infty \rightarrow I = 0$

$$I = \int_0^T |u(y)|\, dy = 0 \longrightarrow |u(y)| = 0 \quad \forall y \in [0, T]$$

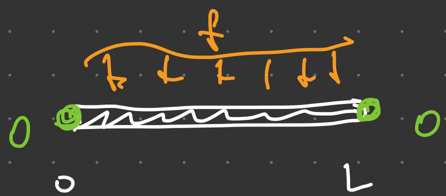
$$\longrightarrow u(y) = 0 \quad \forall y \in [0, T] \xrightarrow[u \text{ is periodic}]{} u(y) = 0 \quad \forall y \in \mathbb{R}.$$

Then, if both methods are applicable, the $u(x)$ is a trivial solution.

3.3 Partial Differential Equations (PDEs)

3.3.1 Heat equation for a finite length bar

Let be $a \in \mathbb{R}^*$, $L \in \mathbb{R}^+$, $f : [0, L] \to \mathbb{R}$, $f \in C^1([0,L])$



$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad x \in]0, L[,\ t > 0$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x)$$ initial condition

BCs: Dirichlet BC

For case $f(x) = 2\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

separation of variables: $u(x,t) = v(x)\, w(t)$

The BCs become:

$$u(0,t) = u(L,t) = v(0)\,w(t) = v(L)\,w(t) = 0$$

I don't want $w(t)=0 \longrightarrow v(0) = v(L) = 0$

$$v(x)\,w'(t) = a^2\, v''(x)\,w(t) \qquad x \in \,]0, L[\,,\ t>0$$

$$\frac{v''(x)}{v(x)} = \frac{1}{a^2}\,\frac{w'(t)}{w(t)} \qquad x \in \,]0, L[\,,\ t>0$$

For this to be true for $\forall x$ and $\forall t$:

$$\frac{v''(x)}{v(x)} = -\lambda = \frac{1}{a^2}\,\frac{w'(t)}{w(t)} \quad ,\ \lambda \in \mathbb{R}\ ,\ \forall x, \forall t$$

$$\left.\begin{aligned} &v''(x) + \lambda v(x) = 0 \qquad x \in \,]0, L[\\ + \text{BCs:}\quad &v(0) = v(L) = 0 \end{aligned}\right\} \longrightarrow \text{Sturm–Liouville problem (ODE)}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad v_n = \sin\left(\frac{n\pi}{L}x\right)$$