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CHAPTER 6: APPLICATIONS OF FOURIER ANALYSIS

TO ODEs AND PDEs

ODE: Ordinary Differential Equation

PDE: Partial Differential Equation.

Formally a system of differential equations, given an interval $]a,b[$, and a function

$$F:]a,b[\times \underbrace{\mathbb{R}^N \times \mathbb{R}^N \times \dots \times \mathbb{R}^N}_{m+1 \text{ times}} \longrightarrow \mathbb{R}^N$$

consist in finding $u:]a,b[\longrightarrow \mathbb{R}^N$ s.t.

$$F(t, u(t), u'(t), \dots, u^{(m)}(t)) = 0$$

If $N=1 \longrightarrow$ ODE, If $N \geq 2 \rightarrow$ PDE

Example:

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \qquad \forall t>0, \quad a_2, a_1, a_0 \in \mathbb{R}$$

$$f(t): \mathbb{R}^+ \longrightarrow \mathbb{R}$$

$$y'(t) = \frac{dy}{dt}(t)$$

ODE

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall x, \forall t>0 \qquad a \in \mathbb{R}.$$

PDE

Note: normally we have more that solution. for such problem: we will add additional data to the problem.

3.2 Ordinary differential equation

3.2.1 Cauchy's problem

Note: $\mathbb{R}^+$: all the real positive

$\mathbb{R}^+ = \{x \in \mathbb{R},\ x > 0\}$

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

Find the solution $u : \mathbb{R}^+ \longrightarrow \mathbb{R}$ of the equation

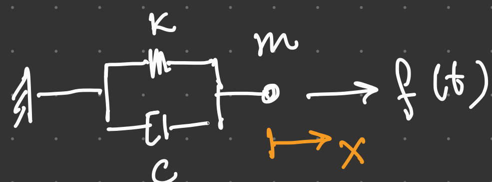
$$a_2 u''(t) + a_1 u'(t) + a_0 u(t) = f(t) \quad \forall t > 0$$

$$u(0) = u_0 \text{ and } u'(0) = u_1 \quad \text{(boundary conditions)}$$

where $a_1, a_0, u_0, u_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}^*$ are given constants and $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is also given and has a "certain" continuity.

Examples:

- An oscillator



$$m x''(t) + C x'(t) + K x(t) = f(t) \quad \forall t \in \mathbb{R}$$

$$x(0) = x_0, \quad x'(0) = x_1 \quad \text{(initial position and velocity)}$$

• Population model

$$p'(t) = (\beta - \delta)\, p(t) \quad \forall t > 0$$

$$p(0) = P_0 \quad \text{and} \quad p'(0) = P_1$$

where $p(t)$ is the population (number of individuals)

$\beta, \delta \in \mathbb{R}$

β: is the birth rate

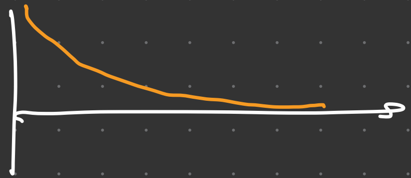
δ: is the death rate

This is just a model: $\beta = 0$ (no new births) and $\delta > 0$

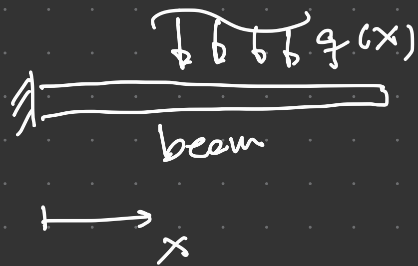
$p'(t) = -\delta\, p(t) \rightarrow$ velocity is negative

$$p(t) = e^{-\alpha t}$$

$$\lim_{t \to \infty} p(t) = 0$$



- Euler–Bernoulli beam



Equilibrium:

$$M''(x) = q(x)$$

where M is the bending moment.

Note: this family of problems will be studiend in Analysis IV using Laplace transform.

3.2.2 Sturm–Liouville problem

let be $u : [0,L] \to \mathbb{R}$ the solution of

$$u''(t) + \lambda u(t) = 0 \qquad \forall\, t \in]0,L[$$

$$u(0) = u(L) = 0 \qquad \text{(homogeneous Dirichlet conditions)}$$

- Example:

steady-state 1D Schrödinger equation.

$$\psi''(x) = \frac{2 m V(x)}{\hbar^2} \psi(x)$$

+ Dirichlet boundary conditions

<u>solution:</u>

$$u''(t) + \lambda u(t) = 0$$

$$u(0) = u(L) = 0$$

The trivial solution $u(t) = 0$ is a possible solution, but we are interested in other possible solution.

For which values $\lambda \in \mathbb{R}$ can we find non-trivial solutions?

- Case 1: $\lambda = 0$, the problem becomes:

$$u''(t) = 0 \quad \forall\, t \in]0, L[$$

$$u(0) = u(L) = 0$$

$$\int u'' \longrightarrow u'(t) = u_1, \quad u_1 \in \mathbb{R},$$

$$u(t) = u_1 t + u_0, \quad u_0, u_1 \in \mathbb{R}$$

$$BC \longrightarrow u(0) = u_1 \cdot 0 + u_0 = u_0 = 0 \longrightarrow u_0 = 0$$

$$\searrow u(L) = u_1 \cdot L + 0 = 0 \longrightarrow u_1 = 0$$

I only get the trivial solution.

- Case 2 : $\lambda < 0$

$$u''(t) + \lambda u(t) = 0 \quad \forall t \in]0, L[\quad \text{with } \lambda < 0 \quad (\lambda \in \mathbb{R}^-)$$

$$u(0) = u(L) = 0$$

That possible solutions to $u''(t) + \lambda u(t) = 0$ (for $\lambda < 0$) are in the form

$$u(t) = \frac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda}) \text{ , } u_1 \in \mathbb{R} \text{ is a constant.}$$

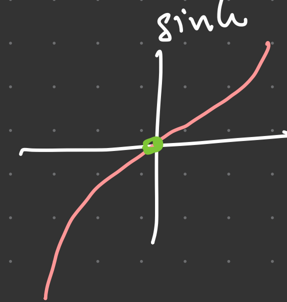
$$u'(t) = \frac{u_1}{\cancel{\sqrt{-\lambda}}} \cancel{\sqrt{-\lambda}} \cosh(t\sqrt{-\lambda}) = u_1 \cosh(t\sqrt{-\lambda})$$

$$u''(t) = \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda})$$

$$u''(t) + \lambda u(t) = \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda}) + \lambda \frac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda})$$

$$= \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda}) - (-\lambda) \frac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda})$$

$$= \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda}) - (\sqrt{-\lambda})^{\cancel{2}} \frac{u_1}{\cancel{\sqrt{-\lambda}}} \sinh(t\sqrt{-\lambda}) = 0 \checkmark$$



let's imposse boundary conditions:

$$u(0) = \frac{u_1}{\sqrt{-\lambda}} \sinh(0\sqrt{-\lambda}) = 0 \checkmark$$

$$u(L) = \frac{u_1}{\sqrt{-\lambda}} \underbrace{\sinh(L\sqrt{-\lambda})}_{\neq 0} = 0 \implies u_1 = 0$$

my solution becomes trivial (again) $u(t) = 0$.

- Case 3: $\lambda > 0$. For this case possible solutions to $u''(t) + \lambda u(t) = 0$ are in the form:

$$u(t) = \frac{u_1}{\sqrt{\lambda}} \sin(t\sqrt{\lambda}) \Rightarrow u'(t) = u_1 \cos(t\sqrt{\lambda})$$

$$\Rightarrow u''(t) = -u_1\sqrt{\lambda} \sin(t\sqrt{\lambda})$$

$$u''(t) + \lambda u(t) = -u_1\sqrt{\lambda} \sin(t\sqrt{\lambda}) + \lambda \frac{u_1}{\sqrt{\lambda}} \sin(t\sqrt{\lambda}) = 0 \checkmark$$

Imposing boundary conditions:

$$u(0) = \frac{u_1}{\sqrt{\lambda}} \underbrace{\sin(0\sqrt{\lambda})}_{=0} = 0 \checkmark$$

$$u(L) = \frac{u_1}{\sqrt{\lambda}} \sin(L\sqrt{\lambda}) = 0 \Rightarrow \sin(L\sqrt{\lambda}) = 0$$

$$\Rightarrow L\sqrt{\lambda} = n\pi, \quad n = 0, 1, \dots$$

Then $\lambda = \left(\frac{n\pi}{L}\right)^2$.

The Sturm-Liouville problem only has non-trivial solutions for $\lambda = \left(\frac{n\pi}{L}\right)^2$ and the solution are:

$$u(t) = u_n \sin\left(\frac{n\pi}{L} t\right)$$ where $u_n \in \mathbb{R}$ is an arbitrary constant