19/11/2024

b) let $f: [0, 2\pi[\longrightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[. \end{cases}$$

extended by 2π-periodicity to $\mathbb{R}$.

$f(x)$ is piecewise defined but $\underline{f \notin C^0(\mathbb{R})}$ (f is not continuous).

$$Ff(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$$

we compute its derivative.

$$\frac{dFf(x)}{dx} = (Ff)'(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \cos((2k+1)x)$$

$$(Ff)'\left(x = \frac{\pi}{2}\right) = \frac{2}{\pi} \sum_{k=0}^{\infty} \cos\left(\frac{2k+1}{2}\pi\right),$$

converges

$k=0 \quad \cos\frac{\pi}{2} = 0$

$k=1 \quad \cos\frac{3\pi}{2} = 0$

$k=2 \quad \cos\frac{5\pi}{2} = 0$

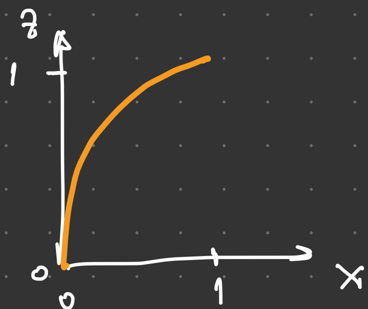
$$(Ff)'(x=\pi) = \frac{2}{\pi}\sum_{k=0}^{\infty} \cos\left((2k+1)\pi\right) \Rightarrow \begin{cases} k=0 & \cos\pi=-1 \\ k=1 & \cos 3\pi=-1 \\ k=2 & \cos 5\pi=-1 \end{cases}$$

$$= \frac{2}{\pi}\sum_{k=0}^{\infty}(-1) \longrightarrow -\infty$$ <u>diverges!</u>

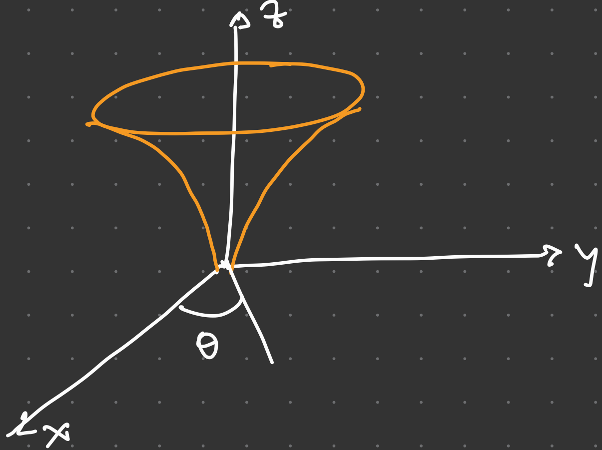
Exercise 1 (Ex 7.2 page 89).
Verify Stokes' theorem for

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=z^4,\ 0 \le z \le 1\} \text{ and } F(x,y,z) = (x^2y, z, x).$$

$y=0$



$y=0, \quad x^2 = z^4$



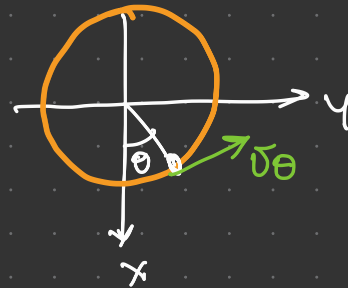
$$\sigma(\theta, z) = (z^2\cos\theta,\ z^2\sin\theta,\ z)$$

$$A = [0, 2\pi] \times [0, 1]$$

$$\sigma_\theta \times \sigma_z = (-z^2\sin\theta, z^2\cos\theta, 0) \times (2z\cos\theta, 2z\sin\theta, 1)$$

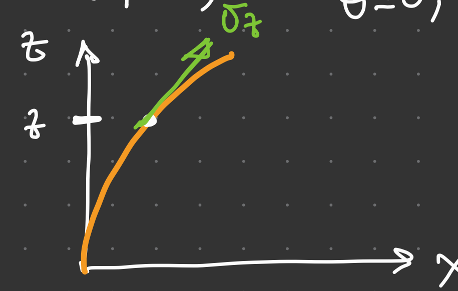
$$= (z^2\cos\theta, z^2\sin\theta, -2z^3)$$

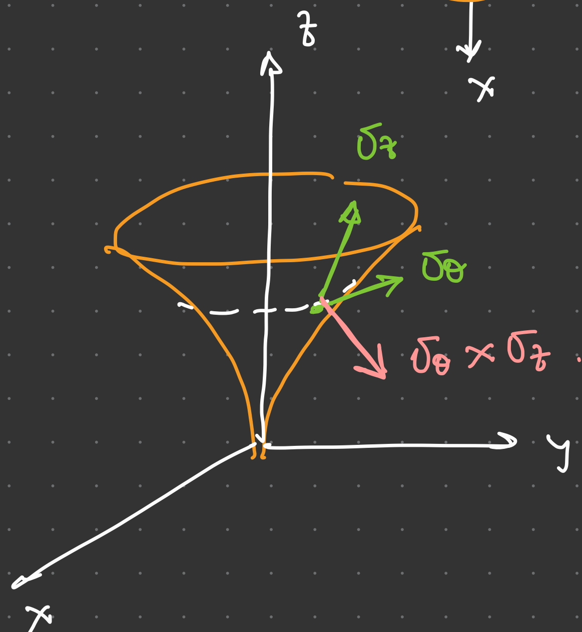
$z = \text{constant}$



$\theta = 0 \quad (y=0)$

$\theta = 0, \quad (2z, 0, 1)$





Stokes:

$$\iint_\Sigma \text{curl}\, F \cdot dS = \int_{\partial\Sigma} F \cdot dl$$

$$\iint_A \text{curl}\, F(\sigma(\theta, z)) \cdot (\sigma_\theta \times \sigma_z)\, d\theta\, dz$$

$$= \int_{\partial\Sigma} F \cdot dl$$

- Theorem 2:

Let $f:\mathbb{R}\to\mathbb{R}$ be a T-periodic s.t., f and f' are piecewise defined. Let

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi n}{T}x\right) + b_n\sin\left(\frac{2\pi n}{T}x\right)\right)$$

be its Fourier series. Then $\forall\, x_0$ and $x\in[0,T]$ we have

$$\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x}\frac{a_0}{2}\,dt + \sum_{n=1}^{\infty}\int_{x_0}^{x}\left(a_n\cos\left(\frac{2\pi n}{T}t\right) + b_n\sin\left(\frac{2\pi n}{T}t\right)\right)dt$$

1.4 other Fourier series formulations

On how to build Fourier series of non-periodic functions.

1.4.1 Fourier cosine series

- Theorem:

  Let $f : [0, L] \to \mathbb{R}$ be a continuous function s.t. f' is piecewise defined. Then, the series

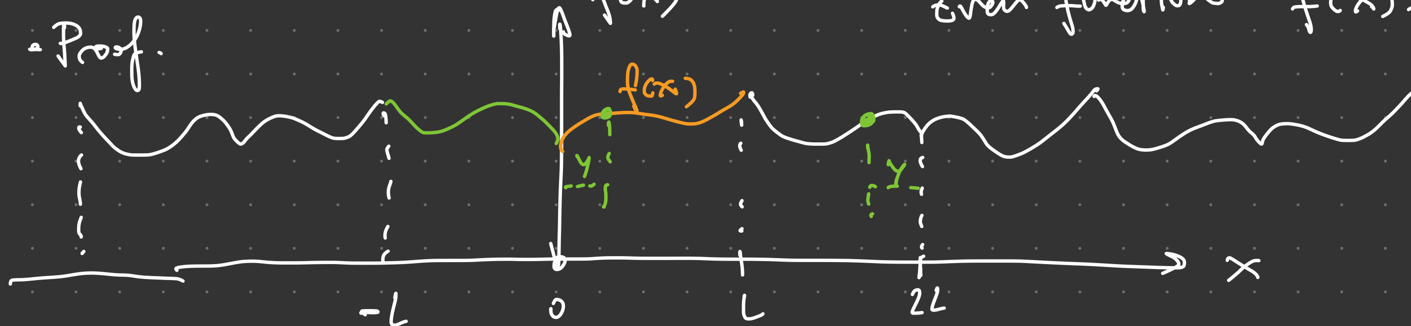
$$F_c = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right) \quad \text{with}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) \quad \text{for } n = 0, 1, \ldots$$

  it is called Fourier cosine series of f. and it converges to f in the interval $[0, L]$. I.e., $f(x) = F_c f(x) \ \forall x \in [0, L]$

• Proof.

Even function $f(x) = f(-x)$



$$g(x) = \begin{cases} 1\text{- extend as even function to } [-L, 0] \\ 2\text{- extend by } 2L \text{ periodicity.} \end{cases}$$

We denote as $g : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic and continuous.

s.t. g' is piecewise-defined.

$$Fg(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right)$$

$g(x)$ is an even function $b_n = 0$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right)$$

$$a_n = \frac{2}{T}\int_0^T g(x)\cos\left(\frac{2\pi n}{T}x\right)dx \overset{T=2L}{=} \frac{1}{L}\int_0^{2L} g(x)\cos\left(\frac{\pi n}{L}x\right)dx$$

$$= \frac{1}{L}\int_0^{L} g(x)\cos\left(\frac{\pi n}{L}x\right)dx + \frac{1}{L}\int_L^{2L} g(x)\cos\left(\frac{\pi n}{L}x\right)dx$$

$$\int_L^{2L} g(x)\cos\left(\frac{\pi n}{L}x\right)dx = -\int_L^{0} \underbrace{g(2L-y)}_{g(y)}\cos\underbrace{\left(\frac{\pi n}{L}(2L-y)\right)}_{\cos\left(\frac{n\pi}{L}y\right)}dy$$

$$x = 2L-y$$

$$dx = -dy$$

$$= -\int_L^{0} g(y)\cos\left(\frac{n\pi}{L}y\right)dy = \int_0^{L} g(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$a_n = 2 \frac{1}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

in $[0, L]$ $g(x) = f(x)$

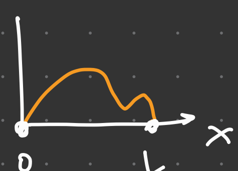
in $[0, L]$

$$f(x) = g(x) = F g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right) = F_c f(x) \quad \square$$

1.4.2 Fourier sine series

- Theorem:

Let $f : [0, L] \to \mathbb{R}$ be a continuous function s.t. $f(0) = f(L) = 0$ and f' is piecewise defined. Then



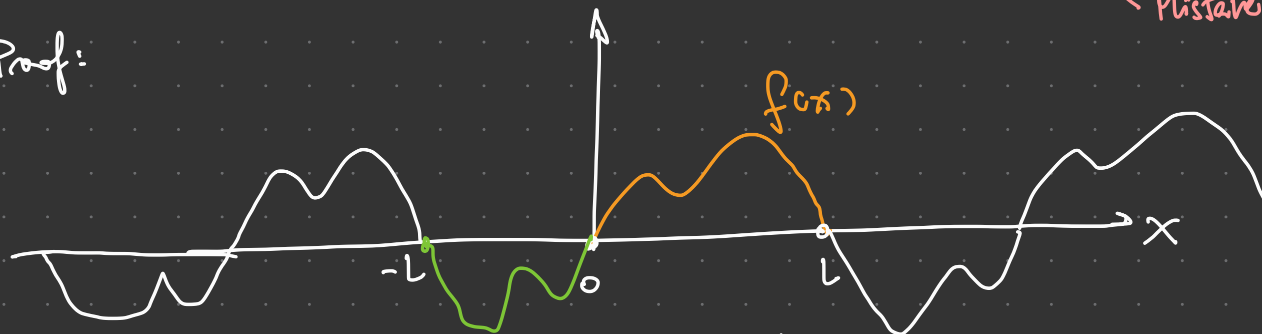
$$F_S f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right) \text{ with}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx \quad \text{for } n \geq 1.$$

denoted as the Fourier sine series of f' and it converges to f in $[0, L]$. So, we have $f(x) = F_s f(x)$ $\forall x \in [0, L]$.

This should be f. Mistake in lecture

Proof:



$$g(x) = \begin{cases} \text{1- Extend } f(x) \text{ as an odd function to } [-L, 0] \\ \text{2- extend by } 2L\text{- periodicity.} \end{cases}$$

$g(x)$ is continuous and $2L$- periodic.

$$F g(x) = g(x) \quad \forall x \in \mathbb{R}$$

in $[0, L]$ $f(x) = g(x)$

$$Fg(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right). \quad T = 2L.$$

$g(x)$ is odd $\nearrow$

$a_n = 0$

$$\text{with } b_n = \frac{2}{2L} \int_0^{2L} g(x) \sin\left(\frac{2\pi n}{2L} x\right) dx$$

$$= \frac{1}{L} \int_0^{2L} g(x) \sin\left(\frac{\pi n}{L} dx\right) dx = \frac{1}{L} \int_{-L}^{L} g(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{1}{L} \int_{-L}^{0} g(x) \sin\left(\frac{\pi n}{L} x\right) dx + \frac{1}{L} \int_0^{L} g(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$\int_{-L}^{0} g(x) \sin\left(\frac{\pi n}{L} x\right) dx = -\int_{L}^{0} \underbrace{g(-y)}_{-g(y) \text{ because } g \text{ is odd}} \overbrace{\sin\left(-\frac{\pi n}{L} y\right)}^{-\sin\left(\frac{\pi n}{L} x\right)} dy$$

$y = -x$

$dy = -dx$

$$= -\int_{L}^{0} g(x) \sin\left(\frac{\pi n}{L} x\right) dx = \int_0^{L} g(x) \sin\left(\frac{\pi n}{L} x\right) dx$$

$$b_n = 2\,\frac{1}{L}\int_0^L g(x)\sin\left(\frac{\pi n}{L}x\right)dx$$

Then $\forall x \in [0, L]$ $f(x) = g(x) = F_S g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$

$$= F_S f(x) \quad \square.$$