$T = 2\pi$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\, x^2\Big|_0^{2\pi} = \frac{4\pi^2}{2\pi} = 2\pi.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$x \cos x^2$

$u = x$

$v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\times \underbrace{\frac{\sin nx}{n}\Big|_0^{2\pi}}_{=0} - \frac{1}{n\pi}\underbrace{\int_0^{2\pi}\sin nx\,dx}_{=0} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$u = x$

$v = -\frac{\cos nx}{n}$

$$= -\frac{1}{\pi}\times \frac{\cos nx}{n}\Big|_0^{2\pi} + \frac{1}{n\pi}\underbrace{\int_0^{2\pi}\cos(nx)\,dx}_{=0}$$

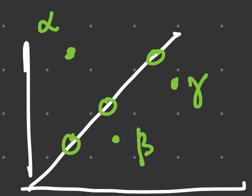
$$= -\frac{1}{\pi}\frac{2\pi}{n} = \frac{-2}{n},$$

$$Ff(x) = \begin{cases} \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 0 \\ \frac{1}{2}(x + x) = x = f(x) & \text{if } x \in \,]0, 2\pi[\\ \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 2\pi. \end{cases}$$

Example 3:

$$f(x) = \begin{cases} \alpha & x = 1/2 \\ \beta & x = 1 \\ \gamma & x = 3/2 \\ x & \text{otherwise} \end{cases}$$

$\alpha, \beta, \gamma \in \mathbb{R}$.



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1.2.5 Complex notation for Fourier series

The Fourier series of T-periodic piecewise-defined function $f: \mathbb{R} \to \mathbb{R}$ in complex form is:

$$Ff(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} x} \quad \text{where } c_n \in \mathbb{C}$$

and are computed as $c_n = \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi n}{T} x} dx$

Indeed, assuming $T = 2\pi$, for the sake of simplicity, we have:

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

$$\cos nx = \frac{1}{2} \left(e^{inx} + e^{-inx} \right), \quad \sin x = \frac{i}{2} \left(e^{-inx} - e^{inx} \right)$$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

Euler

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} \left(e^{inx} + e^{-inx} \right) - \frac{b_n i}{2} \left(e^{inx} - e^{-inx} \right) \right)$$

$$= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \left[\underbrace{\frac{a_n - ib_n}{2}}_{c_n} e^{inx} + \underbrace{\frac{a_n + ib_n}{2}}_{c_{-n}} e^{-inx} \right]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \underbrace{e^{i0x}}_{=1} dx = 2c_0.$$

And for $n \geq 1$

$$\frac{a_n - ib_n}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx\,dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\underbrace{(\cos nx - i\sin nx)}_{e^{-inx}}dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = C_n$$

$$\frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx + i\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\underbrace{(\cos nx + i\sin nx)}_{e^{inx}}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx = C_{-n}$$

$$Ff(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx} \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=-\infty}^{-1} c_n e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Two options for c_n

1. • compute a_0, a_n, b_n as we know.
   • $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$

2. $$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \quad \text{for } T = 2\pi$$

$$\left(c_n = \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi n}{T} x} dx \right.$$

1.3 Properties of Fourier Series

1.3.1 Periodicity and parity

- Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic function such that f and f' ($f' = \frac{df}{dx}$) are piecewise defined. Then:

Example $f(x) = \cos x$

a) Its Fourier series $\mathcal{F}f(x)$ is also T-periodic.

b) If f is an even function (i.e. $f(-x) = f(x)\ \forall x \in \mathbb{R}$)

we have $b_n = 0\ \forall n \geq 1$ and

$$\mathcal{F}f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right)$$

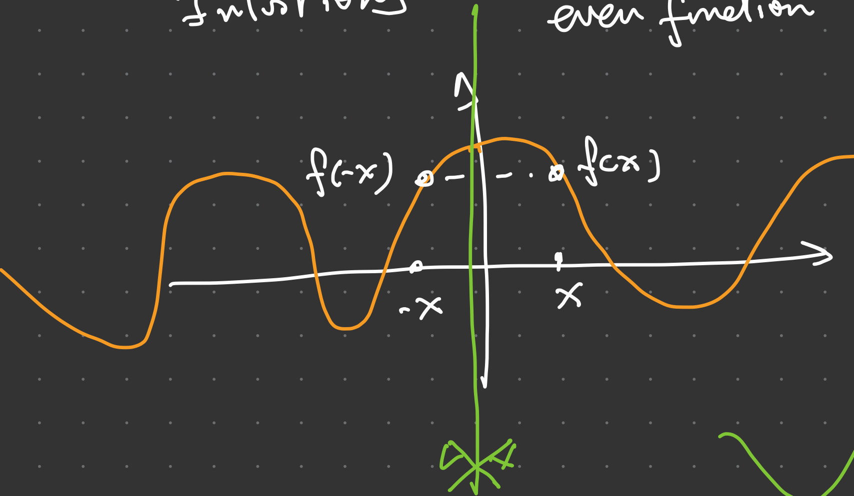
Example $f(x) = \sin(x)$

c) If f is an odd function (i.e., $f(-x) = -f(x)\ \forall x \in \mathbb{R}$)

we have $a_n = 0\ \forall n \geq 0$ then $\mathcal{F}f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T} x\right)$

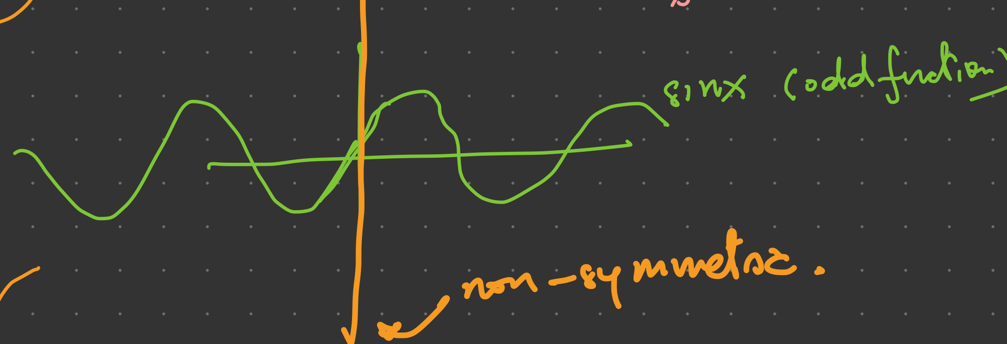
Intuitions

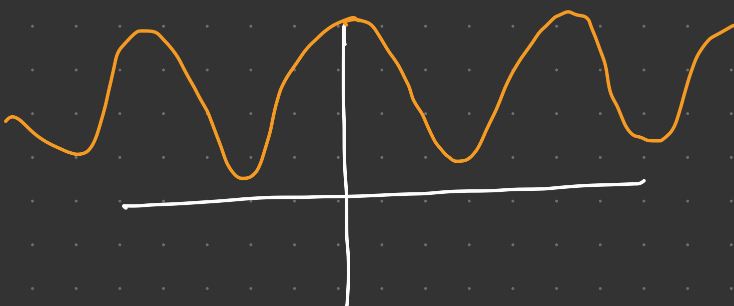
even function



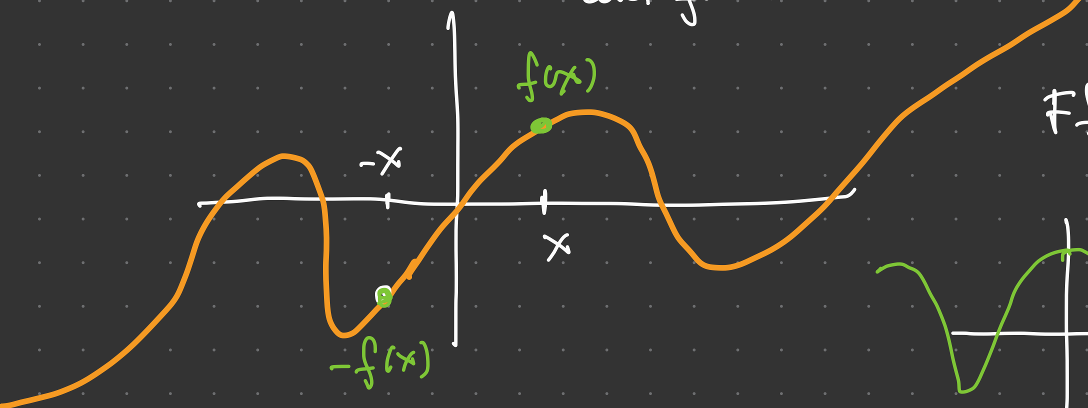
$$F f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{T} x + \cancel{\sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{T} x}$$

$b_n = 0$



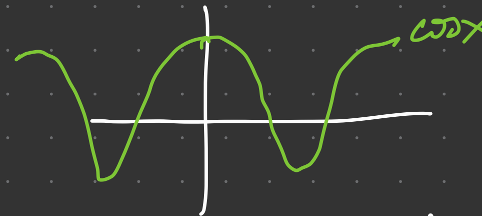


odd function



$$a_n = 0 \quad \forall n \geq 0.$$

$$Ff(x) = \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{T} x$$



Proof: for the sake of simplicity, we assume that $T = 2\pi$.

a) let $n \in \mathbb{N}^*$. The partial Fourier series of order N is:

$$F_N f(x+2\pi) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big(a_n \overbrace{\cos(n(x+2\pi))}^{=\cos nx} + b_n \underbrace{\sin(n(x+2\pi))}_{\sin nx} \Big)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = F_N f(x)$$

$$F f(x+2\pi) = \lim_{N\to\infty} F_N f(x+2\pi) = \lim_{N\to\infty} F_N f(x) = F f(x).$$

b) We have:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$y=-x$

$dy=-dx$

(only for the 1st integral)

$$= -\frac{1}{\pi}\int_{\pi}^{0} \underbrace{f(-y)}_{f(y)} \overbrace{\sin(-ny)}^{-\sin ny}dy + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$f(y)$ because f is an even function

$$= \frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy + \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = 0. \quad \square$$

c) We have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad \forall\, n \geq 0$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$y=-x$

$dy=-dx$

$$= -\frac{1}{\pi}\int_{\pi}^{0} \underbrace{f(-y)}_{-f(y)}\,\overbrace{\cos(-ny)}^{\cos ny}\,dy + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= +\frac{1}{\pi}\int_{\pi}^{0} f(y)\cos ny\,dy + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = 0$$

$\square$