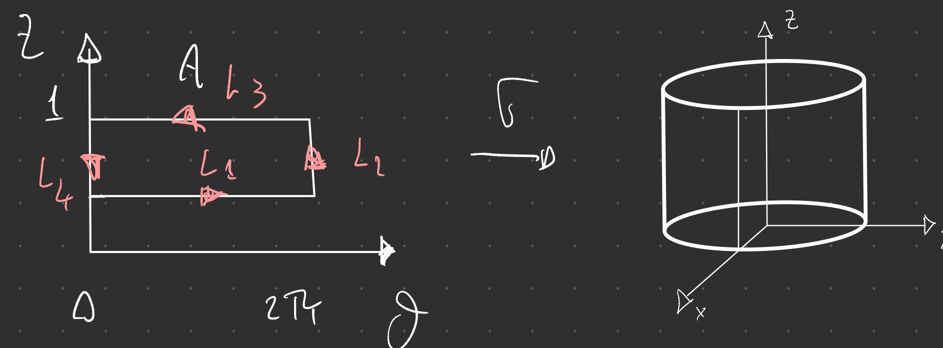
23/10/2024

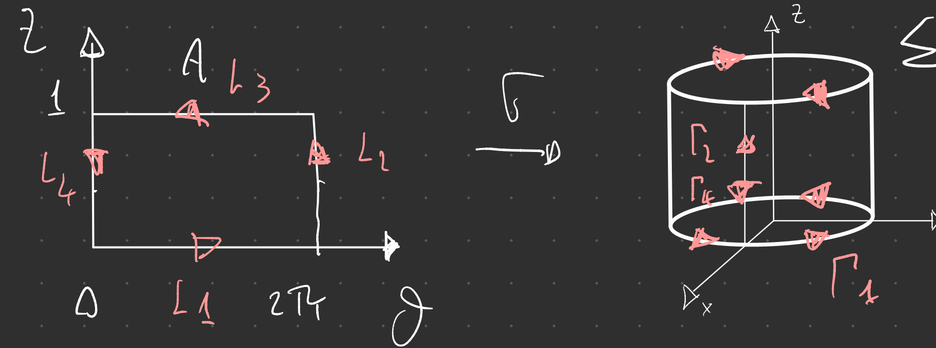
Example 1 : cylinder $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \, , \, 0 \leq z \leq 1\}$

Parametrization $A = \, ]0, 2\pi[ \times ]0, 1[$

$$\Sigma = \{\sigma(\vartheta, z) = (\cos\vartheta, \sin\vartheta, z) : (\vartheta, z) \in \bar{A}\}$$




$$\sigma(\partial A) = \sigma(L_1 \cup L_2 \cup L_3 \cup L_4) = \sigma(L_1) \cup \sigma(L_2) \cup \sigma(L_3) \cup \sigma(L_4)$$




$$\Gamma_1 = \{ \gamma_1(\vartheta) = \sigma(\vartheta, 0) = (\cos\vartheta, \sin\vartheta, 0) \text{ with } \vartheta : 0 \to 2\pi \}$$

$$\Gamma_2 = \{ \gamma_2(z) = \sigma(2\pi, z) = (1, 0, z) \text{ with } z : 0 \to 1 \}$$

$$\Gamma_3 = \{ \gamma_3(\vartheta) = \sigma(\vartheta, 1) = (\cos\vartheta, \sin\vartheta, 1) \text{ with } \vartheta : 2\pi \to 0 \}$$

$$\Gamma_4 = \{ \gamma_4(z) = \sigma(0, z) = (1, 0, z) \text{ with } z : 1 \to 0 \}$$

$\Gamma_1$ circulated counter-clockwise

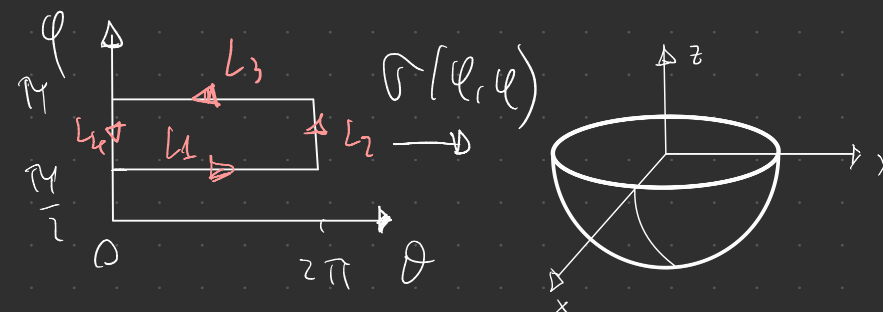
$$\partial\Sigma = \Gamma_1 \cup \Gamma_3$$

$\Gamma_3$ circulated clockwise seen from above

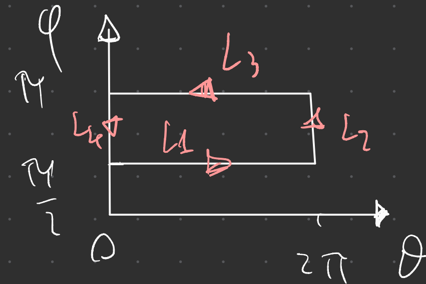
Example 2 semi-sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2=1 \text{ and } z \le 0\}$

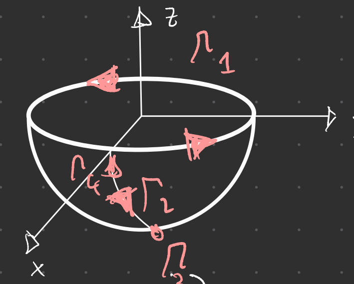
Parameterization $= \,]0, 2\pi[ \times ]\frac{\pi}{2}, \pi[$

$$\Sigma = \{\sigma(\vartheta, \varphi) = (\sin\varphi\cos\vartheta, \sin\varphi\sin\vartheta, \cos\varphi) : (\vartheta, \varphi) \in \bar{A}\}$$




$$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \quad \text{with } \Gamma_i = \sigma(L_i) \text{ with } i = 1..4$$




$$\Gamma_1 = \left\{ \gamma_1(\vartheta) = \sigma\left(\vartheta, \tfrac{\pi}{2}\right) = (\cos\vartheta, \sin\vartheta, 0) \text{ with } \vartheta: 0 \to 2\pi \right\}$$

$$\Gamma_2 = \left\{ \gamma_2(\varphi) = \sigma(2\pi, \varphi) = (\sin\varphi, 0, \cos\varphi) \text{ with } \varphi: \tfrac{\pi}{2} \to \pi \right\}$$

$$\Gamma_3 = \left\{ \gamma_3(\vartheta) = \sigma(\vartheta, \pi) = (0, 0, -1) \text{ with } \vartheta: 2\pi \to 0 \right\}$$

$$\Gamma_4 = \left\{ \gamma_4(\varphi) = \sigma(0, \varphi) = (\sin\varphi, 0, \cos\varphi) \text{ with } \varphi: \pi \to \tfrac{\pi}{2} \right\}$$

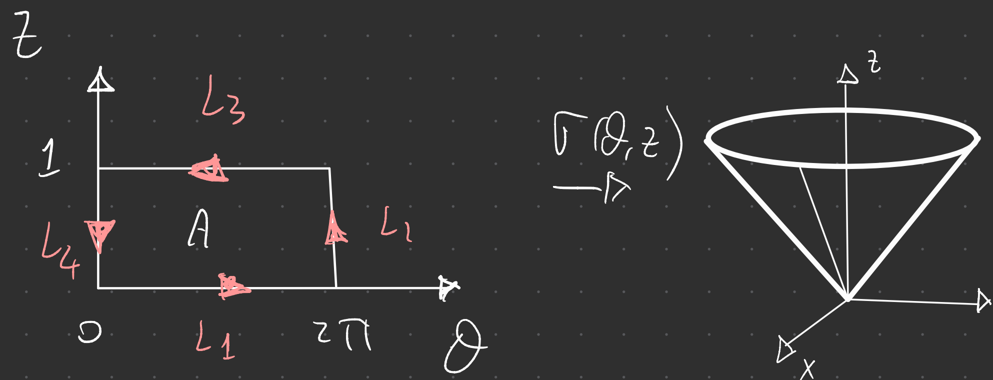
$\partial\Sigma = \Gamma_1$ circulated in counter-clockwise sense

(seen from above)

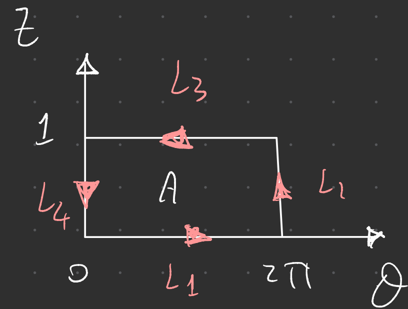
Example 3: cone $\Sigma = \{(x,y,z)\in\mathbb{R}^3 \; z^2 = x^2+y^2 \text{ and } 0\le z\le 1\}$

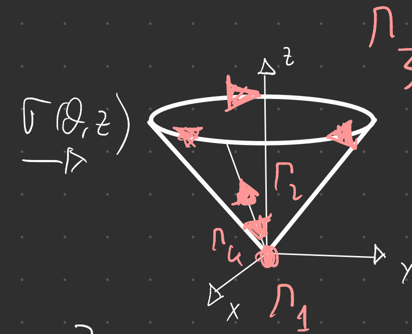
Parameterization: $A = ]0, 2\pi[ \times ]0,1[$

$$\Sigma = \{\sigma(\vartheta,z) = (z\cos\vartheta,\, z\sin\vartheta,\, z) : (\vartheta,z)\in\bar{A}\}$$




$$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \text{ with } \Gamma_i = \sigma(L_i) \text{ for } i = 1..4$$




$$\Gamma_1 = \{ \gamma_1(\theta) = r(\theta, 0) = (0, 0, 0) \text{ with } \theta : 0 \to 2\pi \}$$

$$\Gamma_2 = \{ \gamma_2(z) = r(2\pi, z) = (z, 0, z) \text{ with } z : 0 \to 1 \}$$

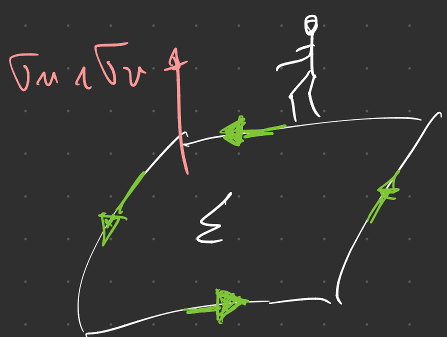
$$\Gamma_3 = \{ \gamma_3(\theta) = r(\theta, 1) = (\cos\theta, \sin\theta, 1) \text{ with } \theta : 2\pi \to 0 \}$$

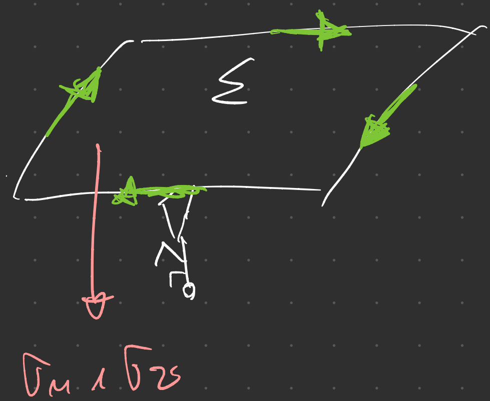
$$\Gamma_4 = \{ \gamma_4(z) = r(0, z) = (z, 0, z) \text{ with } z : 1 \to 0 \}$$

$\partial\Sigma = \Gamma_3$ circulated in clockwise sense

(seen from above)

Practical rule for obtaining the boundary of a surface induced by a parametrization $\sigma(u,v)$: an observer walks along $\partial\Sigma$, with his/her head pointing in the sense of the normal $\sigma_u \wedge \sigma_v$, has the surface on their left



### 3.4.3 Stokes' theorem

- Theorem: let $\Sigma \subset \mathbb{R}^3$ be a piecewise regular surface and orientable. Let $F: \Sigma \to \mathbb{R}^3$ be a vector field such that $F \in C^1(\Sigma, \mathbb{R}^3)$ defined $\to$

$$\left( F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \right)$$

then:

$$\iint_{\Sigma} \operatorname{curl} F \cdot ds = \int_{\partial\Sigma} F \cdot dl$$

- Remarks:

1) Stokes' theorem is the generalization of Green's theorem for vector fields with values in $\mathbb{R}^3$

2) Once chosen the parameterization $\sigma : \underset{(u,v)}{\bar{A}} \to \underset{(x,y,z)}{\Sigma}$

we take $\sigma_u \wedge \sigma_v$ as the normal vector in the surface integral. I.E.

$$\iint_{\Sigma} \operatorname{curl} F \cdot dS = \iint_{A} \left[ \operatorname{curl} F(\sigma(u,v)) \cdot (\sigma_u \wedge \sigma_v) \right] du\, dv$$