15/10/2024

3.2.3 Vector field integrals

- **Definition** let $\Sigma \subset \mathbb{R}^3$ be an orientable regular surface parameterized by: $\sigma : \bar{A} \to \Sigma$, $(u,v) \mapsto \sigma(u,v)$ and let

$$F : \Sigma \to \mathbb{R}^3$$
$$x \mapsto F(x) = (F^1(x), F^2(x), F^3(x))$$ be $F \in C^0(\Sigma, \mathbb{R}^3)$

The integral of F over Σ in the direction of $\sigma_u \times \sigma_v$ is

$$\iint_\Sigma F \cdot ds = \iint_A \left(F(\sigma(u,v)) \cdot (\sigma_u \times \sigma_v) \right) du\, dv$$

using the unit normal $\nu(u,v) = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$:

$$\iint_{\Sigma} F \cdot ds = \iint_{A} \left(F(\sigma(u,v)) \cdot \nu(u,v) \right) \|\sigma_u \times \sigma_v\| \, du\, dv$$

- Remarks:

  - Analogy with the integral of a vector field G along Γ

$$\gamma : [a,b] \longrightarrow \Gamma \qquad\qquad G : \Gamma \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \gamma(t) \qquad\qquad x \longmapsto G(x)$$

$$\int_{\Gamma} G \cdot dl = \int_a^b G(\gamma(t)) \cdot \gamma'(t) \, dt$$

  - For a piecewise regular surface $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ we define:

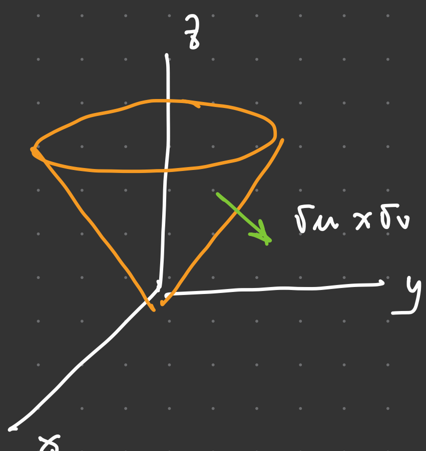
$$\iint_{\Sigma} F \cdot ds = \sum_{i=1}^{k} \iint_{\Sigma_i} F \cdot ds$$

- The integral $\iint_\Sigma F \cdot dS$ computes the <u>flux</u> of F through the surface Σ in the direction of $\sigma_u \times \sigma_v$.

3.2.4 Examples:

- Example 1: let $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } 0 \le z \le 1 \}$ and $F : \Sigma \to \mathbb{R}^3$ defined by $F(x,y,z) = (y, -x, z^2)$

Compute $\iint_\Sigma F \cdot dS$ along the <u>ascending direction</u>



$A =]0, 2\pi[\times]0, 1[$,

$\sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$

$$\sigma_\theta \times \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$

$$\iint_\Sigma F\cdot dS = -\iint_A (F(\sigma(\theta,z))\cdot(\sigma_\theta\times\sigma_z)\,d\theta\,dz$$

$$= +\iint_A (F(\sigma(\theta,z))\cdot(\sigma_z\times\sigma_\theta)\,d\theta\,dz$$

$$= -\int_0^{2\pi}\int_0^1 (z\sin\theta,\ -z\cos\theta,\ z^2)\cdot(z\cos\theta,\ z\sin\theta,\ -z)\,d\theta\,dz$$

$$= -\int_0^{2\pi}\int_0^1 z^2(\cancel{\sin\theta\cos\theta} - \cancel{\cos\theta\sin\theta} - z)\,dz\,d\theta = \int_0^{2\pi}\int_0^1 z^3\,dz\,d\theta$$

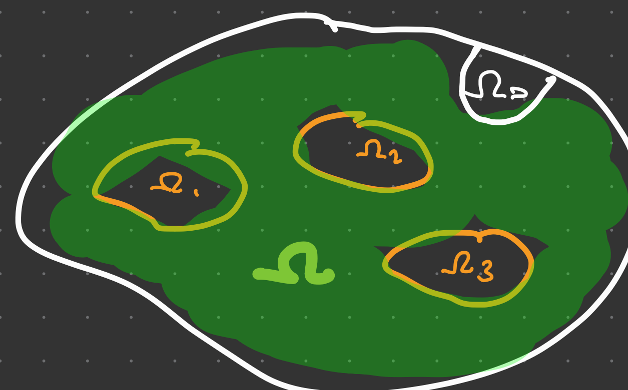
$$= \frac{\pi}{2}$$

3.3 Divergence Theorem

3.3.1 Notation and preliminary notations

• Definition: we say that $\Omega \subset \mathbb{R}^3$ is a regular domain if $\exists$ bounded open domains s.t. $\Omega_0, \Omega_1, \ldots, \Omega_m \subset \mathbb{R}^3$

- $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \bar{\Omega}_j$



- $\bar{\Omega}_j \subset \Omega_0$, $\forall j = 1, 2, \ldots, m$
- $\bar{\Omega}_i \cap \bar{\Omega}_j = 0$ if $i \neq j$ and $i, j = 1, \ldots, m$
- $\partial \Omega_j = \Sigma_j$, $j = 0, 1, \ldots, m$ where Σ_j are (piecewise) orientable surfaces and s.t $\partial \Sigma_j = 0$.

3.3.2 Divergence theorem

- Theorem: let $\Omega \subset \mathbb{R}^3$ be a reguler domain and $\nu : \partial\Omega \to \mathbb{R}^3$ an outer unit normal vector field. let $F : \overline{\Omega} \to \mathbb{R}^3$ be a vector field s.t. $F \in C^1(\overline{\Omega}, \mathbb{R}^3)$ then:

$$\iiint_\Omega \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \iint_{\partial\Omega} (F \cdot \nu)\,dS$$

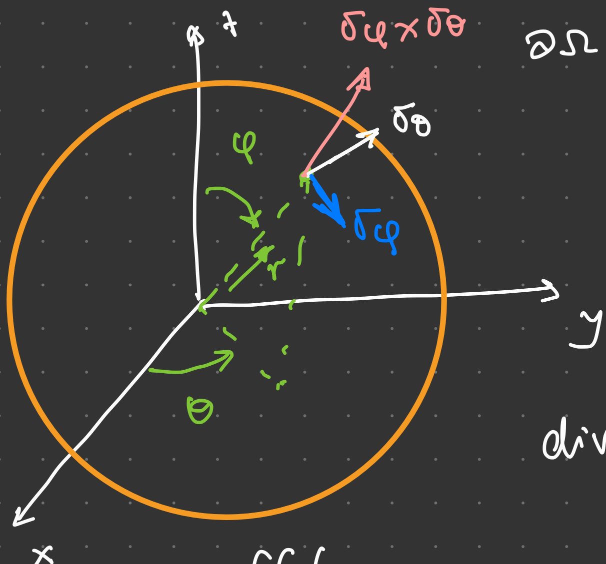
in other words:

$$\iiint_\Omega \left(\frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z} \right) dx\,dy\,dz = \iint_{\partial\Omega} (F^1\nu^1 + F^2\nu^2 + F^3\nu^3)\,dS$$

3.3.3 Examples

- Example 1: verify the divergence theorem for

$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1 \}$ and $F(x,y,z) = (xy, y, z)$.



$\partial\Omega = \Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$

:= shell of the sphere

$$\iiint_\Omega \operatorname{div} F(x,y,z)\, dx\, dy\, dz$$

$\operatorname{div} F = y + 2$

$$\iiint_\Omega (y+2)\, dx\, dy\, dz = \int_0^1 \int_0^{2\pi} \int_0^{\pi} (r \sin\varphi \sin\theta + 2)\, r^2 \sin\varphi \; dr\, d\theta\, d\varphi$$

spher. coords

$$= \cancel{\int_0^1 r^3\,dr \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{=0} \int_0^{\pi} \sin^2\varphi\, d\varphi} + 2\int_0^1 r^2\,dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= \frac{2}{3}\, 2\pi \Big[-\cos\varphi\Big]_0^{\pi} = \frac{8\pi}{3}$$

- Calculation of $\displaystyle\iint_{\partial\Omega} (F\cdot\nu)\,dS$

spherical coordinates with $r=1$

Parameterization of $\partial\Omega = \Sigma$, $\sigma(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$\sigma_\varphi \times \sigma_\theta = (+\sin^2\varphi\cos\theta, +\sin^2\varphi\sin\theta, +\sin\varphi\cos\varphi)$ this an <u>outer normal</u> of Ω

$$\nu = \frac{r_\varphi \times r_\theta}{\| r_\varphi \times r_\theta \|} \qquad \iint_{\partial\Omega} \underbrace{(F \cdot \nu)}_{\text{a scalar field}}\, dS$$

$$= \iint_A (F\cdot\nu)(r(\varphi,\theta))\, \cancel{\| r_\varphi \times r_\theta \|}\, d\varphi\, d\theta$$

$$\underbrace{(F\cdot\nu)}_{g}(r(\varphi,\theta)) = g(r(\varphi,\theta)) = F(r(\varphi,\theta))\cdot\nu(\varphi,\theta)$$

$$= (\sin^2\varphi\cos\theta\sin\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)\cdot \frac{(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)}{\cancel{\|(\quad,\quad,\quad)\|}}$$

$$\iint_{\partial\Omega} F\cdot\nu\, dS = \int_0^{2\pi}\int_0^{\pi} (\sin^2\varphi\cos\theta\sin\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)\ \cdot$$

$$(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\sin^4\varphi\cos^2\theta\sin\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi\right)d\varphi\, d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta}_{=0}\int_0^{\pi}\sin^4\varphi\, d\varphi + \underbrace{\int_0^{2\pi}\sin^2\theta}_{=\pi}\ \underbrace{\int_0^{\pi}\sin^3\varphi\, d\varphi}_{4/3} + 2\pi\underbrace{\int_0^{\pi}\sin\varphi\cos^2\varphi\, d\varphi}_{=2/3}$$

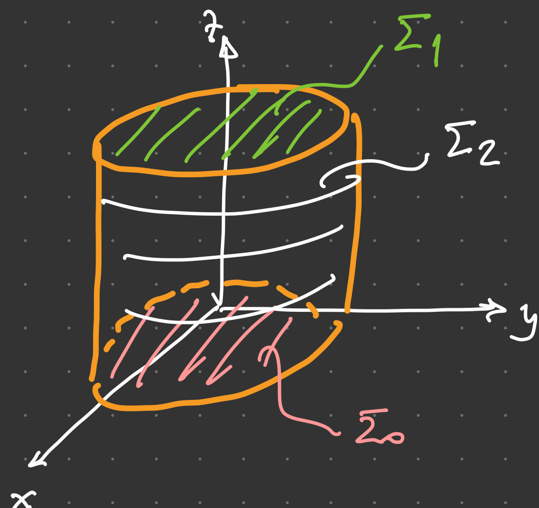
integr. by parts.

$$= \frac{8\pi}{3} = \iiint_\Omega \operatorname{div} F\, dx\, dy\, dz \qquad \square$$

• Example 2: Verify Div. theorem for

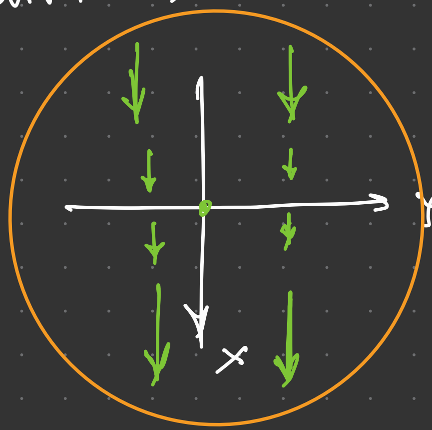
$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2<1 \text{ and } 0 \le z \le 1 \}$

and $F(x,y,z) = (x^2, 0, 0)$



$$\iiint_\Omega \operatorname{div} F \, dV = \iint_{\partial\Omega} F \cdot \nu \, dS$$

$\operatorname{div} F = 2x$



$$\iiint_\Omega \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \iiint_\Omega 2x\,dx\,dy\,dz =$$

$$= \int_0^{2\pi}\int_0^1\int_0^1 \underbrace{2r\cos\theta}_{2x}\; r\,dr\,d\theta\,dz = \underbrace{\int_0^{2\pi} \cos\theta\,d\theta}_{=0} \int_0^1 r^2\,dr \int_0^1 dz = 0$$

$$\iint_{\partial\Omega} F\cdot\nu\,dS = \iint_{\Sigma_0}(F\cdot\nu)\,dS + \iint_{\Sigma_1}(F\cdot\nu)\,dS + \iint_{\Sigma_2}(F\cdot\nu)\,dS$$

Parameterizations: $\Sigma_0 : \sigma^0(\theta, r) = (r\cos\theta, r\sin\theta, 0)$,

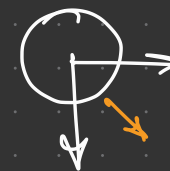
$$A_0 = \,]0, 2\pi[\, \times \,]0, 1[$$

$$\sigma^0_\theta \times \sigma^0_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$$ pointing down (outer normal)

$\Sigma_1 : \sigma^1(\theta, r) = (r\cos\theta, \sin\theta, 1)$, $A_0 = A_1$, $\sigma^1_\theta \times \sigma^1_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$ point down (inner)

$\Sigma_2 : \sigma^2(\theta, z) = (\cos\theta, \sin\theta, z)$, $A_2 =]0, 2\pi[\times]0, 1[$

$$\sigma^2_\theta \times \sigma^2_z = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$ (outer normal)



$$\iint_{\Sigma_0} + \iint_{\Sigma_1} + \iint_{\Sigma_2}$$

be careful with the sign here.