Chapter 1 : Differential operators.

1.1.2 Recalls, notations, and terminology.

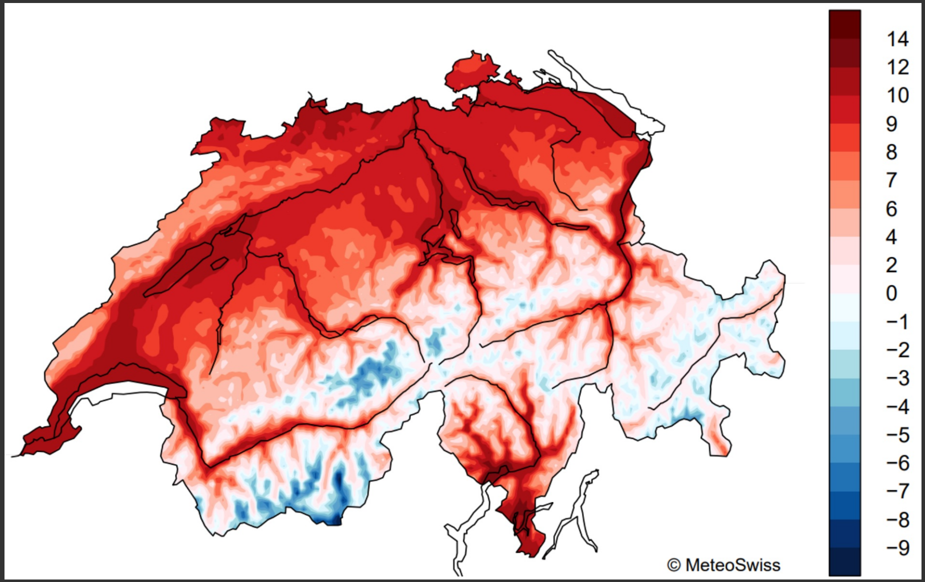
- $\mathbb{N}$: natural numbers $(\{0, 1, 2, \dots\})$
- $\mathbb{Z}$: integer $(\dots, -2, -1, 0, 1, 2, 3, \dots$
- For $n=2$ $\quad (x_1, x_2) = (x, y)$, for $n=3$ $\quad (x_1, x_2, x_3) = (x, y, z)$

  $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$, $\quad n > 0$.
- A function that has real values defined on $\Omega \subset \mathbb{R}^n$

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = f(x_1, x_2, \dots, x_n)$$

that depends on multiple variables $(x_1, x_2, \dots, x_n)$

is called a <u>scalar field</u> defined on Ω.



- For $k \in \mathbb{N}$ we write that $f \in C^k(\Omega)$ if all the derivatives with order $\leq k$ exist and are continuous in Ω

- A function that has vector real values and is defined in $\Omega \subset \mathbb{R}^n$

$$F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad n > 0,\ m > 1$$

$$x \longmapsto F(x) = (F_1(x), F_2(x), \dots, F_m(x))$$

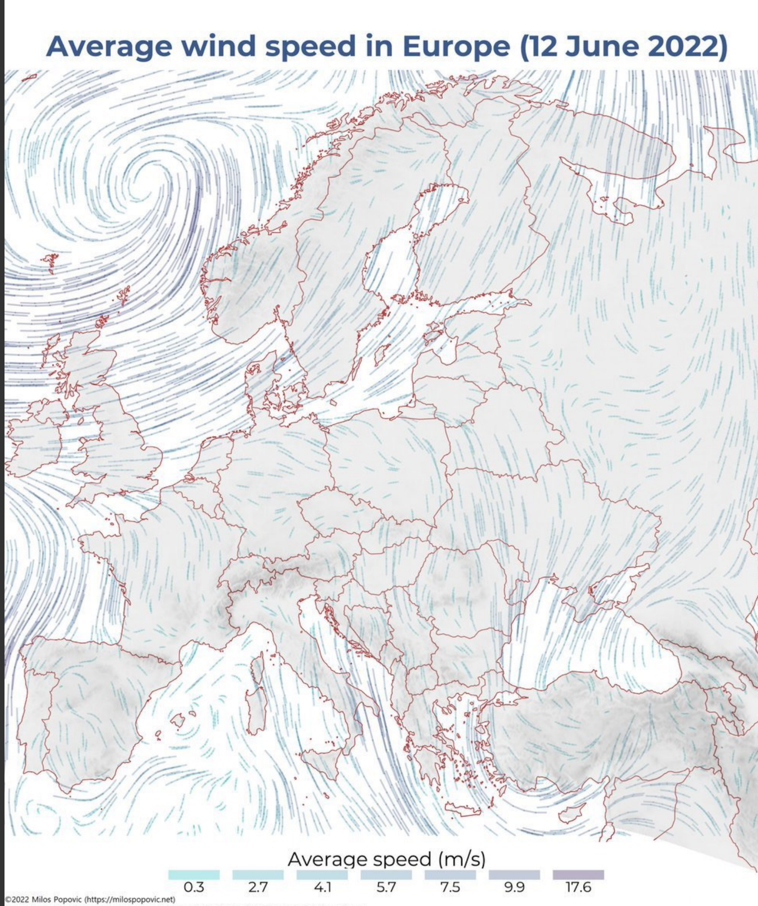
where

$$F_i : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$x \longmapsto F_i(x) = F_i(x_1, x_2, \dots, x_n)$$

$i = 1, \dots, m$

is a <u>vector field</u>. You can see this as a collection of scalar fields $\{F_i\}_{i=1}^{m}$



- For $k \in \mathbb{N}$ we write $F \in C^k(\Omega, \mathbb{R}^m)$ if $F_i \in C^k(\Omega)$, for $i = 1, \dots, m$

- The differential operator nabla, denoted as ∇, is defined by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

1.2 The gradient operator

1.2.1 Definition

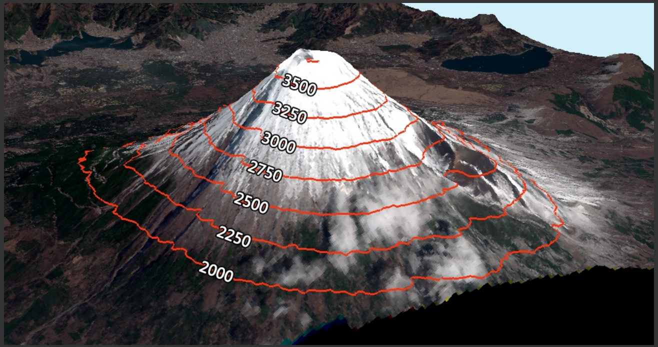
Let be $\Omega \subset \mathbb{R}^n$ an open domain. and $f : \Omega \longrightarrow \mathbb{R}$

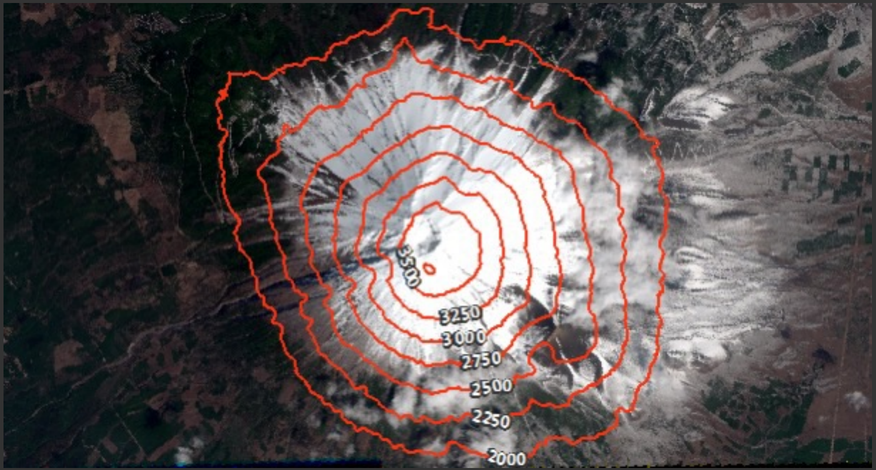
$$x \longmapsto f(x) = f(x_1, \dots, x_n)$$

a scalar field $f \in C^1(\Omega)$. The gradient of f is denoted as grad f (or ∇f, or Df) and is defined as

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

It is the "product" of ∇ and f.





The gradient of a scalar field is a vector field

$$\text{grad} f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

- $f = f(x,y)$ $(\Omega \subset \mathbb{R}^2)$, $\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

  $f = f(x,y,z)$ $(\Omega \subset \mathbb{R}^3)$, $\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$

1.2.2 Examples

- Example 1: let be $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x,y,z) \longmapsto f(x,y,z) = x^2 y^3 \sin(z^2)$$

$$\text{grad} f(x,y,z) = \left(2x y^3 \sin z^2 ,\ 3x^2 y^2 \sin z^2 ,\ 2x^2 y^3 z \cos z^2 \right) \in \mathbb{R}^3$$

- Example 2: let's consider a particle with mass m that is placed at $P = (x,y,z) \in \mathbb{R}^3$ and another particle with mass M and placed at $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$. The potential of the gravitational field is given by

$$f : \mathbb{R}^3 \setminus P_0 \longrightarrow \mathbb{R}$$

$$(x,y,z) \rightarrow f(x,y,z) = \frac{g\, m\, M}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

$$r \stackrel{\text{def}}{=} \|P - P_0\| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

$$f(x,y,z) = \varphi(r) = \frac{c}{r}, \quad c = gmM$$

$$\operatorname{grad} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

$$\frac{\partial f}{\partial x} = \frac{d\varphi}{dr}\frac{\partial r}{\partial x}, \quad \frac{d\varphi}{dr} = -\frac{c}{r^2}$$

$$\frac{\partial f}{\partial y} = \frac{d\varphi}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)$$

$$\frac{\partial f}{\partial z} = \frac{d\varphi}{dr}\frac{\partial r}{\partial z} \qquad = \frac{x-x_0}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y-y_0}{r}, \quad \frac{\partial r}{\partial z} = \frac{z-z_0}{r}$$

$$\frac{\partial f}{\partial x} = \frac{d\psi}{dr} \cdot \frac{\partial r}{\partial x} = -\frac{c}{r^2} \frac{x-x_0}{r} = -c \frac{x-x_0}{r^3}$$

$$\frac{\partial f}{\partial y} = -c \frac{y-y_0}{r^3}, \quad \frac{\partial f}{\partial z} = -c \frac{z-z_0}{r^3}$$

$$\operatorname{grad} f = -\frac{c}{r^3} (x-x_0, y-y_0, z-z_0)$$

1.3 The divergence operator.

let be $\Omega \subset \mathbb{R}^n$ an open domain and $F : \Omega \to \mathbb{R}^n$
$x \mapsto F(x)$

a vector field such that (s.t.) $F \in C^1(\Omega, \mathbb{R}^n)$

The divergence operator of F, denoted $\operatorname{div} F$ (or $\nabla \cdot F$) is defined by

$$\operatorname{div} F(x) = \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) + \cdots + \frac{\partial F_n}{\partial x_n}(x)$$

You can see this as the scalar product of ∇ and F.

Remark: $\operatorname{div} F : \Omega \to \mathbb{R}$ is a scalar field.

1.3.2 Examples:

Example 1: Let be

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto F(x,y,z) = \left(x^2 - e^y, \sin(xz), y^2 e^{2xz}\right)$$

$$\operatorname{div} F(x,y,z) = \frac{\partial}{\partial x}(x^2 - e^y) + \frac{\partial}{\partial y}(\sin(xz)) + \frac{\partial}{\partial z}\left(y^2 e^{2xz}\right)$$

$= 2x + 0 + 2xy^2 e^{2xz} = 2x(1 + y^2 e^{2xz}) \in \mathbb{R}.$

Example 2: let us consider the gravitational force of the prev. ex.

We assume that $P_0 = (0,0,0)$

$$F : \mathbb{R}^3 \setminus (0,0,0) \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto F(x,y,z) = -\frac{c}{r^3}(x,y,z)$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{cx}{r^3}\right) = -c\left(\frac{1}{r^3} - 3\frac{x}{r^4}\frac{\partial r}{\partial x}\right) = c\,\frac{3x^2 - r^2}{r^5}$$

$$\frac{\partial F_2}{\partial y} = \frac{3y^2 - r^2}{r^5}, \quad \frac{\partial F_3}{\partial z} = \frac{3z^2 - r^2}{r^5}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = c\,\frac{3(x^2+y^2+z^2) - 3r^2}{r^5} = c\,\frac{3r^2 - 3r^2}{r^5} = 0$$