$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad V_n = \sin\left(\frac{n\pi}{L} x\right)$$

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$$-\lambda = \frac{1}{a^2}\frac{w'(t)}{w(t)} \longrightarrow w_n'(t) + \left(a\frac{n\pi}{L}\right)^2 w_n(t) = 0 \qquad \forall n > 0$$

$$W_n = \exp\left(-\left(\frac{an\pi}{L}\right)^2 t\right) \cdot c \quad \text{with } C \in \mathbb{R}^*$$

$$u(x,t) = V(x)W(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right) e^{-a^2\left(\frac{n\pi}{L}\right)^2 t} \cdot C$$

$$= \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right) e^{-a^2\left(\frac{n\pi}{L}\right) t}$$

abuse of notation

$$u(x,0) = f(x)$$

$$\left.\begin{array}{l} u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right)\cdot 1 = f(x) \\ F_s(f(x)) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi}{L}x\right) \end{array}\right\} \Rightarrow \alpha_n = \beta_n$$

$$\beta_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n}{L}x\right)dx = \alpha_n$$

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{\pi n}{L}x\right) e^{-a^2\left(\frac{n\pi}{L}\right)^2 t}$$

$$f(x) = 2\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) \longrightarrow \beta_1 = 2,\ \beta_3 = -1$$

$$\beta_2 = 0,\ \beta_n = 0\ \forall n > 3.$$

$$u(x,t) = 2\sin\left(\frac{\pi}{L}x\right)e^{-a^2\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi}{L}x\right)e^{-a^2\left(\frac{3\pi}{L}\right)^2 t}$$

Note:

$u(x,y,z,t)$: the temperature at point $(x,y,z) \in \Omega$ at instant t.

$$\frac{\partial u}{\partial t}(x,y,z,t) = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = a^2 \Delta u(x,y,z,t)$$

- Heat equation in 3D.

- steady-state situation ($t \longrightarrow \infty$)

$\rightarrow$ $a^2 \Delta u(x,y,z) = 0 \longrightarrow$ laplacian or laplace problem.

$\rightarrow$ $a^2 \Delta u(x,y,z) = f(x,y,z) \longrightarrow$ Poisson problem.

- Heat equation in an infinite length bar:

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t), \qquad \forall x \in \mathbb{R}, \ \forall t \in \mathbb{R}^+$$

$$u(x,0) = f(x)$$

$\rightarrow$ This can be solve using Fourier transform.

3.3.2 <u>Wave equation in a finite length bar:</u>

Let be $c, L \in \mathbb{R}^+$, $f, g : [0, L] \to \mathbb{R}$, $f, g \in C^3([0,L])$

s.t. $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$, find the solution $u(x,t)$ of:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad x \in]0, L[, \quad t > 0.$$

$$u(0,t) = u(L,t) = 0 \qquad t > 0$$

$$u(x,0) = f(x) \qquad x \in]0, L[$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad x \in]0, L[$$

For the particular case $f = 0$ and $g(x) = \sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{2\pi}{L}x\right)$

$$u(x,t) = v(x)w(t)$$

$$v(x)\,w''(t) = c^2\, v''(x)\, w(t) \qquad x \in]0, L[, \quad t > 0.$$

$$v(0)\,w(t) = v(L)\,w(t) = 0 \qquad t > 0$$

$$\Longrightarrow v(0) = v(L) = 0$$

$$\frac{V''(x)}{V(x)} = -\lambda = \frac{1}{c^2}\,\frac{W''(t)}{W(t)}$$

Sturm-Liouville problem: $\lambda = \left(\frac{n\pi}{L}\right)^2$ with $n > 0$.

$$V_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

$$W_n''(t) + \left(c\,\frac{n\pi}{L}\right)^2 W_n(t) = 0 \longrightarrow \text{ODE.}$$

$$W_n(t) = a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right]}_{W_n(t)} \underbrace{\sin\left(\frac{n\pi}{L} x\right)}_{V_n(x)}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left[a_n \cdot 1 + b_n \cdot 0\right] \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x) \longrightarrow \text{Fourier in sines.}$$

$$a_n = \frac{2}{L}\int_0^L f(y)\sin\left(\frac{n\pi}{L}y\right)dy \quad \left(f=0 \rightarrow a_n = 0\right)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} n b_n \frac{\pi c}{L} \underbrace{\cos\left(\frac{n\pi c}{L}\overset{t=0}{t}\right)}_{1} \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{n=1}^{\infty} n b_n \frac{\pi c}{L} \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

$$\sum_{n=1} n b_n \sin\left(\frac{n\pi}{L}x\right) = \frac{L}{\pi c} g(x) \longrightarrow \text{Fourier series in sines.}$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(y)\sin\left(\frac{n\pi}{L}y\right)dy$$

$$b_n = 0 \text{ for } n > 2$$

$$g(x) = \sin\left(\frac{n\pi}{L}x\right) - \sin\left(\frac{2\pi}{L}x\right) \longrightarrow b_1 = \frac{L}{\pi c}, \quad b_2 = -\frac{L}{2\pi c}$$

$a_n = 0 \quad \forall n$

$$u(x,t) = \frac{L}{\pi c} \sin\left(\frac{\pi c}{L} t\right) \sin\left(\frac{\pi}{L} x\right) - \frac{L}{2\pi c} \sin\left(\frac{2\pi c}{L} t\right) \sin\left(\frac{2\pi}{L} x\right)$$