CHAPTER 2: FOURIER TRANSFORM

2.1 Introduction

2.1.1 Motivation

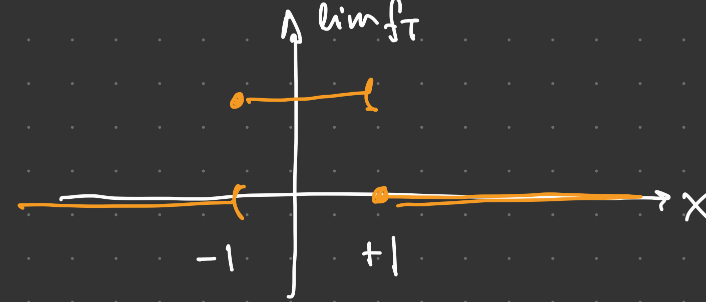
Fourier series allows to develop periodic functions as an infinite series of trigonometric functions. Fourier transform allows to study general functions (not necessarily periodic).

- Idea: let $T > 0$ and f_T be a T-periodic function defined as

$$f_T = \begin{cases} 0 & \text{if } x \in [-T/2, 1[\\ 1 & \text{if } x \in [-1, 1[\\ 0 & \text{if } x \in [1, T/2[\end{cases}$$

When $T \to \infty$ we have

$$\lim_{T\to\infty} f_T \begin{cases} 1 & \text{if } x \in [-1, 1[\\ 0 & \text{if } x \notin [-1, 1[\end{cases}$$



The general idea is to consider general functions as limits of periodic functions when $T \to \infty$.

2.1.2 Heuristic "discovery" of Fourier transform:

for the sake of simplicity

Let $f_T : \mathbb{R} \to \mathbb{R}$ a continuous T-periodic function s.t. f' is piecewise defined. The Fourier series of f_T in complex notation is:

$$F f_T(y) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} y} \quad \forall y \in \mathbb{R}$$

where $c_n = \frac{1}{T}\int_0^T f_T(x)\, e^{-i\frac{2\pi n}{T}x}\, dx = \frac{1}{T}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\frac{2\pi n}{T}x}\, dx$

Let us introduce $\Delta\alpha = \frac{2\pi}{T}$ and $\alpha_n = n\,\Delta\alpha = \frac{2\pi}{T}n$

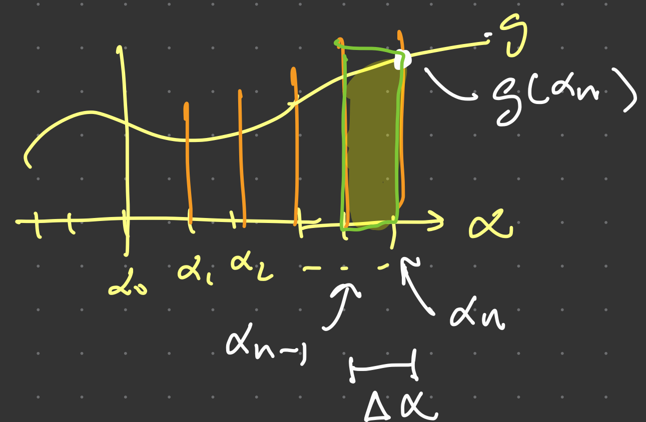
$\frac{1}{T} = \frac{\Delta\alpha}{2\pi}$. $\quad c_n = \frac{\Delta\alpha}{2\pi}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx$

$$Ff_T(y) = \sum_{n=-\infty}^{\infty}\left[\frac{\Delta\alpha}{2\pi}\int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx\right] e^{i\alpha_n y}$$

$$Ff_T(y) = \frac{1}{2\pi}\int_{-T/2}^{T/2} f_T(x)\left[\Delta\alpha \sum_{n=-\infty}^{\infty} e^{-i\alpha_n(x-y)}\right] dx$$

$$\Delta\alpha = \alpha_n - \alpha_{n-1} = \frac{2\pi}{T}n - \frac{2\pi}{T}(n-1)$$

$$(\alpha_n - \alpha_{n-1}) \sum_{n=-\infty}^{+\infty} e^{-i\alpha_n (x-y)}$$ this is a Riemann sum.



$$\Delta\alpha \sum_{n=-\infty}^{+\infty} e^{-i\alpha_n (x-y)} \underset{\text{Riemann sum}}{=} \int_{-\infty}^{+\infty} e^{-i\alpha (x-y)} \, d\alpha$$

$$F f_T(y) = \frac{1}{2\pi} \int_{-T/2}^{T/2} f_T(x) \left[\int_{-\infty}^{+\infty} e^{-i\alpha(x-y)} \, d\alpha \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} f_T(x) e^{-i\alpha x} dx \right] e^{i\alpha y} d\alpha .$$

Because f_T is continuous, then we have $Ff_T(y) = f_T(y)$

$$\iff \lim_{T\to\infty} f_T(y) = \lim_{T\to\infty} Ff_T(y) = f(y)$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx \right]}_{\mathcal{F}(f) \text{ or } \hat{f} \text{ : the Fourier transform of } f.} e^{i\alpha y} d\alpha$$

Then:

$$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx$$

and

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha y}\, d\alpha.$$

2.2 Fourier transform of a function

2.2.1 Definitions:

let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise-defined function s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty \qquad \text{(the integral is bounded).}$$

1) The Fourier transform of f is

$$\mathcal{F}(f) \text{ or } \hat{f} : \mathbb{R} \to \mathbb{C}$$
$$\alpha \mapsto \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx$$

2) The inverse Fourier transform of f:

$$\mathcal{F}^{-1}(f) \text{ or } \hat{f}^{-1} : \mathbb{R} \to \mathbb{C}$$
$$x \mapsto \hat{f}^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\alpha)\, e^{+i\alpha x}\, d\alpha$$