14/11/2024

1.3.2 Parseval's identity

- Theorem: let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic s.t. f and f' are piecewise defined. Then:

$$\frac{2}{T}\int_0^T (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the Fourier coefficients.

Example: let $f: [0, 2\pi[\to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$ and extended by 2π-periodic.

The Fourier coeffs. are $a_0 = 1$, $a_n = 0$ $\forall n \geq 1$ and

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{otherwise} \end{cases}$$

We have:

$$\frac{2}{T}\int_0^T (f(x))^2\,dx = \frac{2}{2\pi}\int_0^{2\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_0^{\pi} (1)^2 = 1.$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{4}{n^2\pi^2} \underset{n=2k+1}{=} \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2}$$

$$1 = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \longrightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

Proof of Parseval's identity: for the sake of simplicity we assume that $T = 2\pi$ and that f is continuous.

Then $f(x) = Ff(x)$ $\forall x \in \mathbb{R}$. and

$$\frac{1}{\pi}\int_0^{2\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,Ff(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right)dx$$

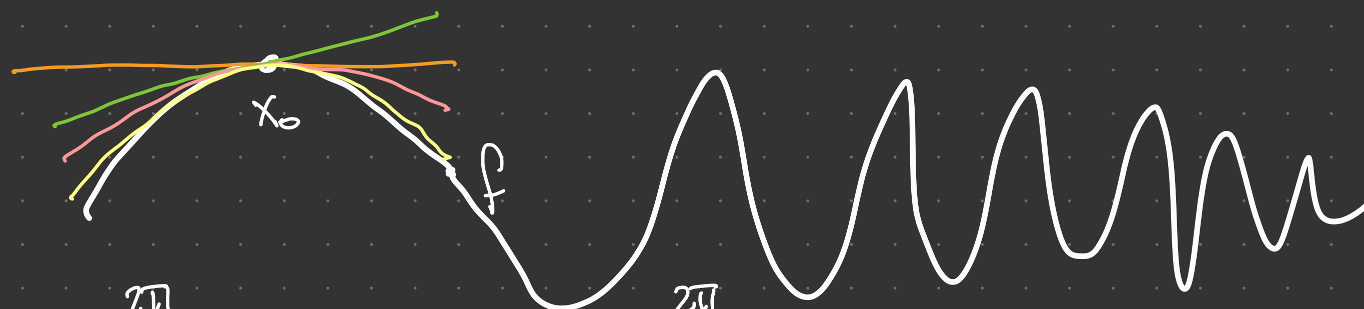
$$= \frac{1}{\pi}\frac{a_0}{2}\underbrace{\int_0^{2\pi} f(x)}_{\pi a_0} + \frac{1}{\pi}\sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi} f(x)\cos nx\,dx}_{= \pi a_n}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi} f(x)\sin nx\,dx}_{\pi b_n}$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \quad \square$$

Note:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \cdots$$



$$a_n = \int_0^{2\pi} f(x) \cos nx \quad , \quad b_n = \int_0^{2\pi} f(x) \sin nx$$

$$\tau = 2\pi$$

1.3.3 Differentiation and integration term by term of Fourier series

- Theorem 1: Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous T-periodic function s.t. f' and f'' are piecewise defined. Let

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)\right)$$

its Fourier series. Then, the series obtained by differentiating $Ff(x)$ term by term converges $\forall x \in \mathbb{R}$ and we have:

$$\frac{dFf(x)}{dx} = \sum_{n=1}^{\infty} \frac{2\pi n}{T}\left(- a_n \sin\left(\frac{2\pi n}{T}x\right) + b_n \cos\left(\frac{2\pi n}{T}x\right)\right)$$

$$= \frac{1}{2}\left(f'(x+0) + f'(x-0)\right) \text{ where}$$

$$f'(x+0) = \lim_{\substack{t \to x \\ t > x}} f'(t), \quad f'(x-0) = \lim_{\substack{t \to x \\ t < x}} f'(t).$$

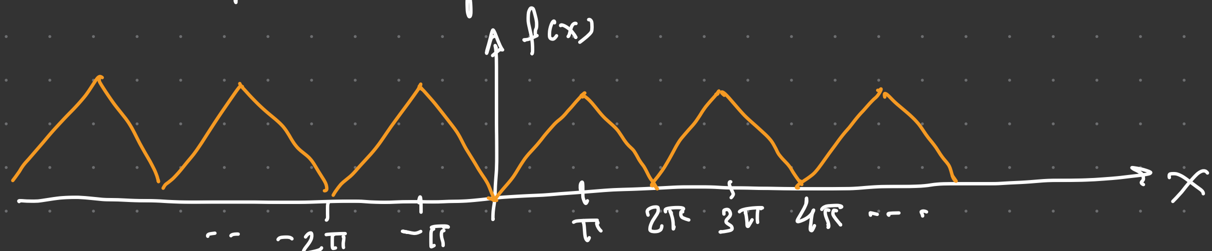
- Comment: As f is continuous, then $Ff(x) = f(x) \ \forall x \in \mathbb{R}$ and, in particular, the series obtained as $\dfrac{dFf(x)}{dx}$ converges to $f'(x) \ \forall x \in \mathbb{R}$ where f' is continuous.

Examples:

a) Let $f : [0, 2\pi[\to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x & \text{if } x \in [0, \pi[\\ 2\pi - x & \text{if } x \in [\pi, 2\pi[\end{cases}$$

is extended by 2π-period. to $\mathbb{R}$.



The Fourier coeffs are $b_n = 0 \quad \forall n \geq 1$, $a_0 = \pi$

$$a_n = \begin{cases} 0 & \text{if } n \geq \text{ is even} \\ -\dfrac{4}{\pi n^2} & \text{if } n \geq \text{ is odd} \end{cases}$$

$$f(x) = Ff(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos\left((2k+1) x \right)}{(2k+1)^2}$$

$\uparrow$

$f \in C^0(\mathbb{R})$

$f'(x)$ is piecewise defined

$$\frac{d\,Ff(x)}{dx} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)x\right)}{2k+1} = \frac{1}{2}\left(f'(x+0) + f'(x-0) \right)$$

$$\begin{cases} \frac{1}{2}(1-1) = 0 & \text{if } x = 0 \\ \frac{1}{2}(1+1) = 1 = f'(x) & \text{if } x \in]0,\pi[\\ \frac{1}{2}(-1+1) = 0 & \text{if } x = \pi \\ \frac{1}{2}(-1-1) = -1 = f'(x) & \text{if } x \in [\pi, 2\pi[\\ \frac{1}{2}(1-1) = 0 & \text{if } x = 2\pi \end{cases}$$