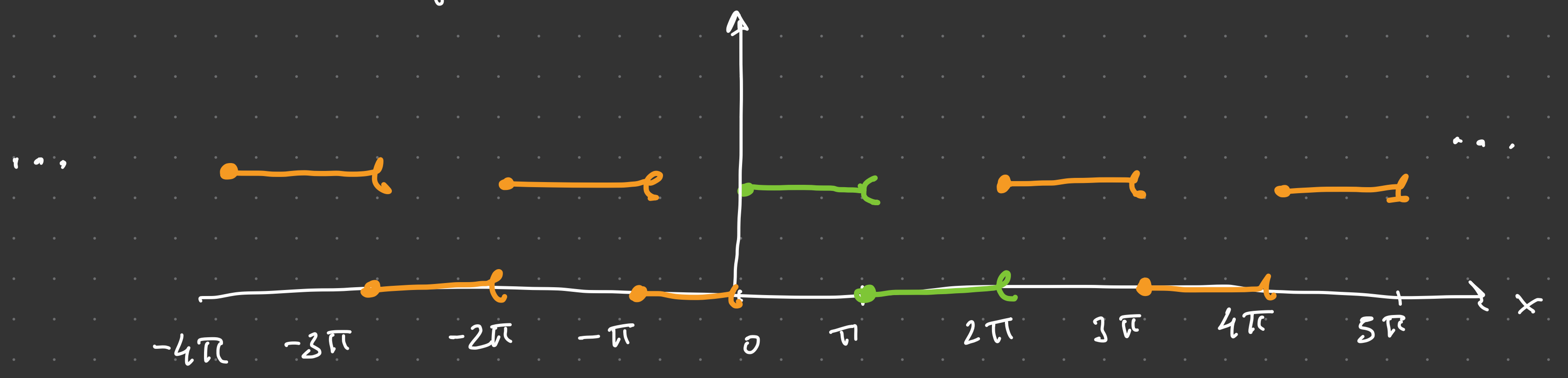
07/11/2024

2.2.4 Examples:

- Example 1: Let $f : [0, 2\pi[\to \mathbb{R}$ defined as
$[0, 2\pi)$

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$

be a function that is extended by 2π-periodicity to $\mathbb{R}$. Compute the Fourier series of f and compare both in $[0, 2\pi]$



$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right)$$

$T = 2\pi$

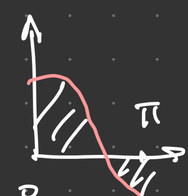
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$T = 2\pi$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) = \frac{1}{\pi} \int_0^{\pi} 1\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cdot dx = 1$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} 1 \cdot \cos nx\, dx + \underbrace{\frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cdot \cos nx\, dx}_{=0}$$

$$= \frac{1}{\pi} \left. \frac{\sin nx}{n} \right|_0^{\pi} = \frac{1}{\pi} (0 - 0)$$



$a_n = 0 \quad \forall n > 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx = \frac{1}{\pi}\int_0^{\pi} 1\cdot \sin nx\,dx + \underbrace{\frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot \sin nx\,dx}_{=0}$$

$$= \frac{-1}{\pi}\,\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{-1}{n\pi}\left(\cos(n\pi) - \cos 0\right) = \frac{-1}{n\pi}\left((-1)^n - 1\right)$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

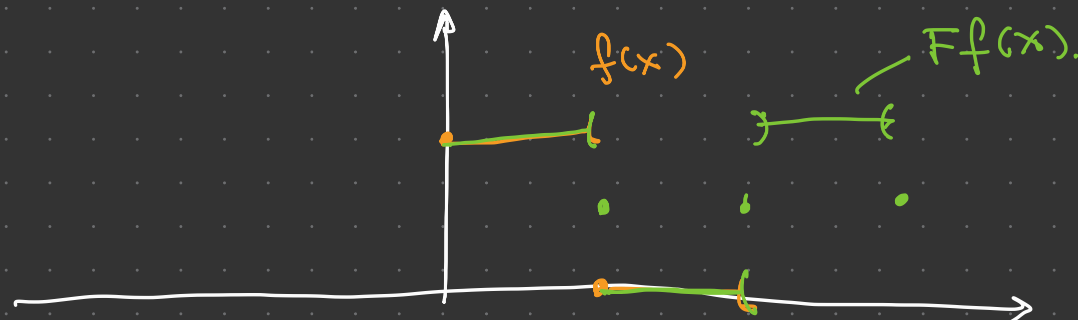
$$Ff(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi}\sin nx = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$$

$n = 2k+1$, $k \in \mathbb{N}$

this is odd $\forall k$

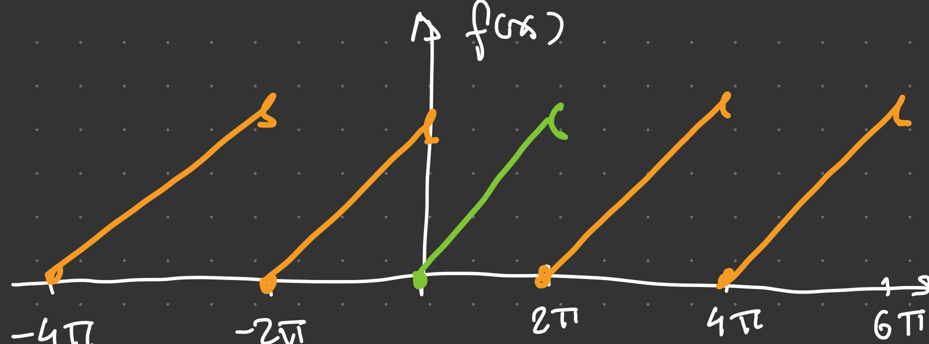
Comparing $Ff(x)$ and $f(x)$ in $[0, 2\pi]$.

$$
Ff(x) = \begin{cases}
\frac{1}{2}\left(f(0+0) + f(0-0)\right) = \frac{1}{2}(1+0) = \frac{1}{2} & \text{if } x = 0 \\
\frac{1}{2}(1+1) = 1 = f(x) & \text{if } x \in]0, \pi[\\
\frac{1}{2}(1+0) = \frac{1}{2} & \text{if } x = \pi \\
\frac{1}{2}(0+0) = 0 = f(x) & \text{if } x \in]\pi, 2\pi[\\
\frac{1}{2}(0+1) = \frac{1}{2} & \text{if } x = 2\pi
\end{cases}
$$



Example 2: Same as example 1 with $f : [0, 2\pi[\longrightarrow \mathbb{R}$ with $f(x) = x$, extended by 2π-periodicity to $\mathbb{R}$.

Compute $Ff(x)$ and compare in $[0, 2\pi]$.



$T = 2\pi$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\,x^2\Big|_0^{2\pi} = \frac{4\pi^2}{2\pi} = 2\pi.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$x\cos x^2$

$u = x$

$v = \dfrac{\sin nx}{n}$

$$= \frac{1}{\pi}\times\underbrace{\frac{\sin nx}{n}\Big|_0^{2\pi}}_{=0.} - \underbrace{\frac{1}{n\pi}\int_0^{2\pi}\sin nx\,dx}_{=0} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$u = x$

$v = -\dfrac{\cos nx}{n}$

$$= -\frac{1}{\pi}\times\frac{\cos nx}{n}\Big|_0^{2\pi} + \underbrace{\frac{1}{n\pi}\int_0^{2\pi}\cos(nx)\,dx}_{=0}$$

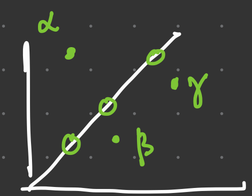
$$= -\frac{1}{\pi}\frac{2\pi}{n} = -\frac{2}{n},$$

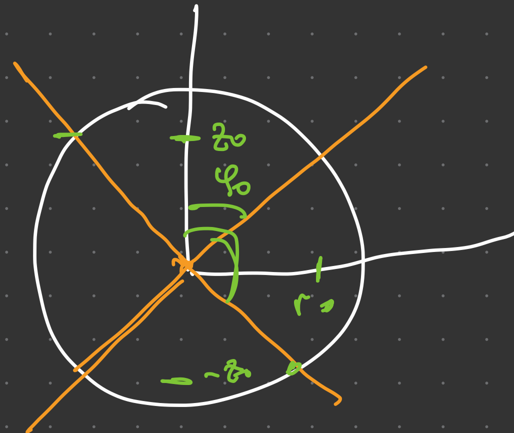
$$Ff(x) = \begin{cases} \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 0 \\ \frac{1}{2}(x + x) = x = f(x) & \text{if } x \in \,]0, 2\pi[\\ \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 2\pi. \end{cases}$$

Example 3:

$$f(x) = \begin{cases} \alpha & x = 1/2 \\ \beta & x = 1 \\ \gamma & x = 3/2 \\ x & \text{otherwise} \end{cases}$$

$\alpha, \beta, \gamma \in \mathbb{R}$.





$$\int_{-z_0}^{z_0} \text{cone}$$

$$\int_0^{2\pi}\int_0^{\varphi_0}$$