31/10/2024.

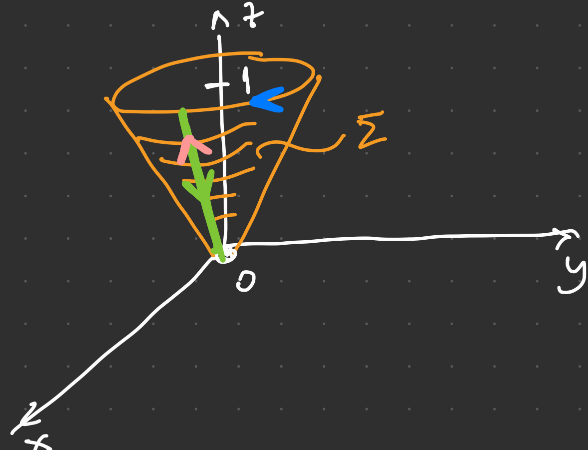
3.4.4 Examples

- Example 1 : verify the stokes' theorem. for

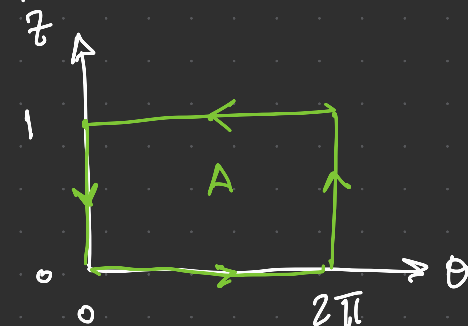
$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2+y^2 \text{ and } 0<z<1\}$$

and $F(x,y,z) = (z,x,y)$.

$$\rightarrow \iint_\Sigma \text{curl}\, F \cdot dS = \iint_A (\text{curl}\, F)(\sigma(\theta,z)) \cdot (\sigma_\theta \times \sigma_z)\, d\theta\, dz.$$



$$\sigma(\theta,z) = (z\cos\theta, z\sin\theta, z)$$

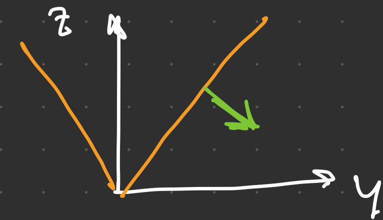


$$\text{curl} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (\text{curl} F)(\sigma(\theta, z)) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$I = \iint_{\Sigma} \text{curl} F \cdot ds = \int_0^{2\pi} \int_0^1 (1, 1, 1) \cdot (\sigma_\theta \times \sigma_z) \, dz \, d\theta .$$

$$\sigma_\theta = (-z \sin\theta, z\cos\theta, 0), \quad \sigma_z = (\cos\theta, \sin\theta, 1)$$

$$\sigma_\theta \times \sigma_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$



$$I = \int_0^{2\pi} \int_0^1 z (\cos\theta + \sin\theta - 1) \, dz \, d\theta = \frac{1}{2} \int_0^{2\pi} (\underbrace{\cos\theta}_{=0} + \underbrace{\sin\theta}_{=0} - 1) \, d\theta = -\pi$$

$$\rightarrow \int_{\partial\Sigma} F\cdot dl$$

$\sigma: (\theta,z) \to (x,y,z)$

$\sigma^{-1}: (x,y,z) \to (\theta,z)$

$\Gamma:$

$$\sigma(\theta,z) = (z\cos\theta,\ z\sin\theta,\ z).$$

$$\gamma(\theta) = (\cos\theta,\ \sin\theta,\ 1)$$

$\theta: 2\pi \longrightarrow 0.$

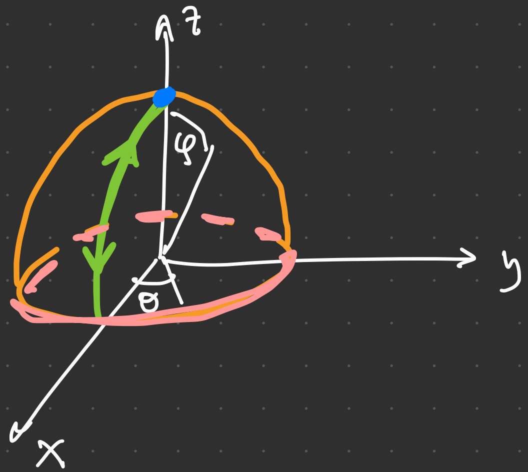
$$\int_{\partial\Sigma} F\cdot dl = \int_{\underset{\alpha}{\Gamma(\sigma^{-1})}} F\cdot dl = \int_{2\pi}^{0} F(\gamma(\theta))\cdot\gamma'(\theta)\,d\theta =$$

$$= \int_{2\pi}^{0} (1,\ \cos\theta,\ \sin\theta)\cdot(-\sin\theta,\ \cos\theta,\ 0)\,d\theta = \int_{2\pi}^{0} (\underbrace{-\sin\theta}_{=0} + \cos^2\theta)\,d\theta$$

$$\left(\int_0^{2\pi} \cos^2\theta = \pi\right) \qquad = -\pi.$$

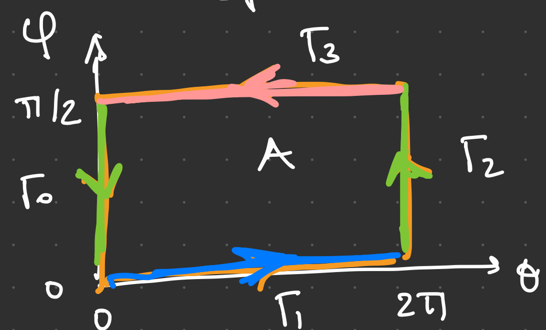
• Example 2 : $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \text{ and } z \geq 0\}$

$F(x,y,z) = (z, x, y^2)$



$\sigma(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

↑ spherical coords.



$\rightarrow \int_{\partial\Sigma} F \cdot dl$

$\Gamma_0 : \theta = 0, \varphi : \pi/2 \to 0$

$\Gamma_1 : \varphi = 0, \theta : 0 \to 2\pi$

$\Gamma_2 : \theta = 2\pi, \varphi : 0 \to \pi/2$

$\Gamma_3 : \varphi = \pi/2, \theta : 2\pi \to 0$

$$\alpha_i = \gamma_0(\sigma^{-1})$$

$$\int_{\partial\Sigma} F \cdot dl = \sum_{i=0}^{3} \int_{\alpha_i} F \cdot dl$$

$$\int_{\alpha_1} F \cdot dl = \int_0^{2\pi} F(\gamma_1(\theta)) \cdot \gamma_1'(\theta)\, d\theta = 0.$$

$$\gamma_1(\theta) = \sigma(\theta, \varphi=0) = (0, 0, 1) , \quad \gamma_1'(\theta) = (0, 0, 0)$$

$$\gamma_0(\varphi) = \sigma(\theta=0, \varphi) = (\sin\varphi, 0, \cos\varphi) , \quad \varphi : \pi/2 \to 0$$

$$\gamma_2(\varphi) = \sigma(\theta=2\pi, \varphi) = (\sin\varphi, 0, \cos\varphi) , \quad \varphi : 0 \longrightarrow \pi/2$$

$$\int_{\alpha_0} F \cdot dl = \int_{\pi/2}^{0} F(\gamma_0(\theta)) \cdot \gamma_0'(\theta)\, d\theta$$

$$\int_{\alpha_2} F \cdot dl = \int_0^{\pi/2} F(\gamma_2(\theta)) \cdot \gamma_2'(\theta)\, d\theta = -\int_{\alpha_0} F \cdot dl$$

$$\gamma_3(\theta) = \sigma(\theta, \varphi = \pi/2) = (\quad , \quad , \quad)$$

$$\int_{\alpha_3} F \cdot dl = \int_{2\pi}^{0} F(\gamma_3(\theta)) \cdot \gamma_3'(\theta)\, d\theta = \ldots = -\pi$$

$$\rightarrow \iint_{\Sigma} \operatorname{curl} F = \ldots = -\pi$$