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3.3.4 Corollary and application

- Corollary: If the domain $\Omega \subset \mathbb{R}^3$ and the unit outer normal field $\nu : \partial\Omega \to \mathbb{R}^3$ then

$$\mathrm{Vol}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu)\, ds = \iint_{\partial\Omega} (G_i \cdot \nu)\, ds \quad , \; i = 1, 2, 3$$

where $F(x,y,z) = (x,y,z)$, $G_1(x,y,z) = (x,0,0)$, $G_2(x,y,z) = (0,y,0)$

$G_3(x,y,z) = (0,0,z)$

Proof: $\operatorname{div} F = 3 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

$\operatorname{div} G_i = 1$

$$\mathrm{Vol}(\Omega) = \iiint_\Omega 1 \, dV = \frac{1}{3} \iint_{\partial\Omega} F \cdot \nu \, dS$$

The same is true for G_i, $i = 1, 2, 3$.

- Practical remark: Given a parameterization of $\partial\Omega$ defined as:

$$\sigma : A \longrightarrow \partial\Omega$$

$$(u, v) \longmapsto \left(\sigma^1(u,v), \sigma^2(u,v), \sigma^3(u,v)\right)$$

then we have

$$\sigma_u \times \sigma_v = \begin{pmatrix} \sigma_u^2 \sigma_v^3 - \sigma_u^3 \sigma_v^2 \\ \sigma_u^3 \sigma_v^1 - \sigma_u^1 \sigma_v^3 \\ \sigma_u^1 \sigma_v^2 \quad \sigma_u^2 \sigma_v^1 \end{pmatrix}$$

this <u>outer</u> normal

$$\mathrm{Vol}(\Omega) = \iint_{\partial\Omega} (G_i \cdot \nu)\, dS = \iint_A G_i(\sigma(u,v)) \cdot \frac{\sigma_u \times \sigma_v}{\cancel{\|\sigma_u \times \sigma_v\|}} \cancel{\|\sigma_u \times \sigma_v\|}\, du\, dv$$

For $i = 1$

$$\mathrm{Vol}(\Omega) = \iint_A \sigma^1(u,v) \left(\sigma^2_u \sigma^3_v - \sigma^3_u \sigma^2_v \right) du\, dv$$

- Application: continuity equation.

Let $v(x,y,z)$ the flow velocity of a fluid at a point (x,y,z) at the instant t.

Let $\rho(x,y,z)$ be the density of the fluid at point (x,y,z) and at instant t.

Mass of the fluid in a domain Ω

$$\text{mass} = \iiint_{\Omega} \rho \, dv$$

$$\iint_{\partial\Omega} \rho \, v \cdot \nu \, ds = -\frac{\partial \, \text{mass}}{\partial t}$$

$$\iiint_{\Omega} \operatorname{div}(\rho v) \, dv = \iint_{\partial\Omega} \rho \, v \cdot \nu \, ds = -\frac{\partial}{\partial t} \iiint_{\Omega} \rho \, dv$$

div theorem.

$$\operatorname{div}(\rho v) = -\frac{\partial}{\partial t} \rho \longrightarrow \operatorname{div}(\rho v) + \frac{\partial}{\partial t} \rho = 0$$

3.4 Stokes' theorem (or Kelvin - Stokes)

3.4.1 Motivation

- Chapter 2: Green's theorem

$$\iint_B \operatorname{curl} G(x,y)\,dx\,dy = \int_{\partial B} G \cdot dl \quad \text{for } B \subset \mathbb{R}^2 \text{ a}$$

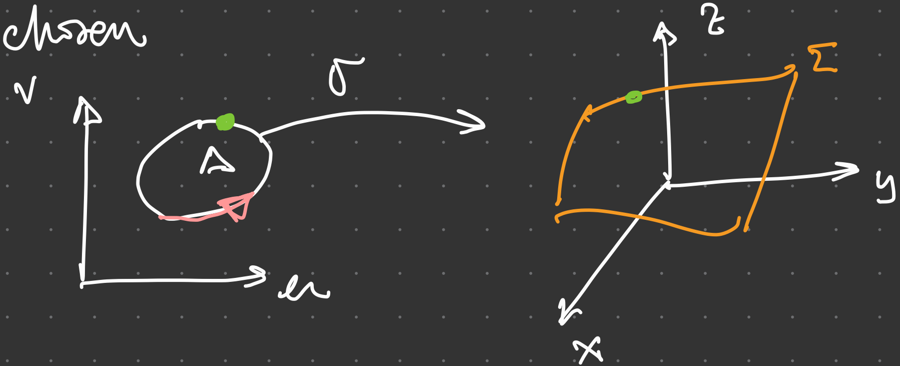
regular domain with boundary ∂B positively oriented and $G : \overline{B} \to \mathbb{R}^2$ s.t. $G \in C^1(B, \mathbb{R}^2)$

- Motivation: to obtain a similar result for a surface $\Sigma \subset \mathbb{R}^3$

$F : \overline{\Sigma} \to \mathbb{R}^3$ s.t. $F \in C^1(\Sigma, \mathbb{R}^3)$

3.4.2 Determination of the boundary of a surface and its sense of circulation

1) If $\Sigma \subset \mathbb{R}^3$ is a regular surface and $\sigma : \bar{A} \to \Sigma$,
then $\partial\Sigma = \sigma(\partial A)$ is independent of the parameterization chosen



2) The sense of circulation of $\partial\Sigma$ is induced by the parameterization σ and is obtained by circulating ∂A in the positive sense.

3) If Σ is piecewise regular then

$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ and we proceed as follows

- we remove from $\sigma(\partial A)$ the curves Γ that reduce to a single point
- we remove the curves that are circulated twice
- what remains is the boundary $\partial\Sigma$