26/09/2024.

2.3.2 Examples (continuation)

Example 2: let be a vector field $\Delta = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$F : \Delta \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto F(x,y) = (F_1(x,y), F_2(x,y)) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

and we consider the domains:

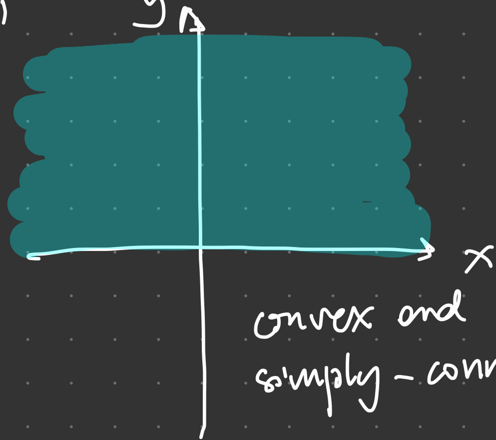
$\Omega_1 = \{ (x,y) \in \mathbb{R}^2 : y > 0 \}$, $\Omega_2 = \{ (x,y) \in \mathbb{R}^2 : y < 0 \}$

$\Omega_3 = \mathbb{R}^2 \setminus \{ (x,y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0 \}$, $\Omega_4 = \mathbb{R}^2 \setminus \{(0,0)\}$

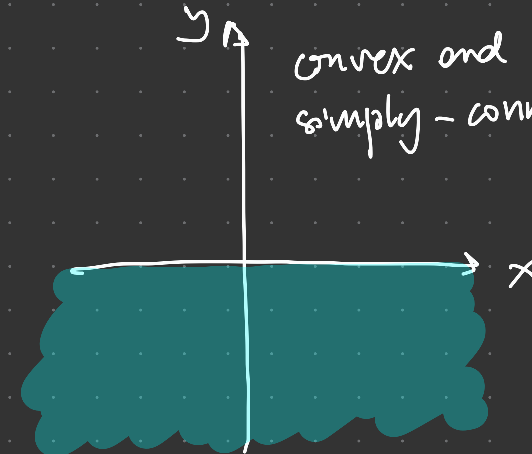
Does F derive from a potential in Ω_i for $i = 1,2,3,4$?

If so, find the potential. Otherwise, find a closed curve $\Gamma \subset \Omega_i$ s.t. $\int_\Gamma F \cdot dl \neq 0$.

Ω_1



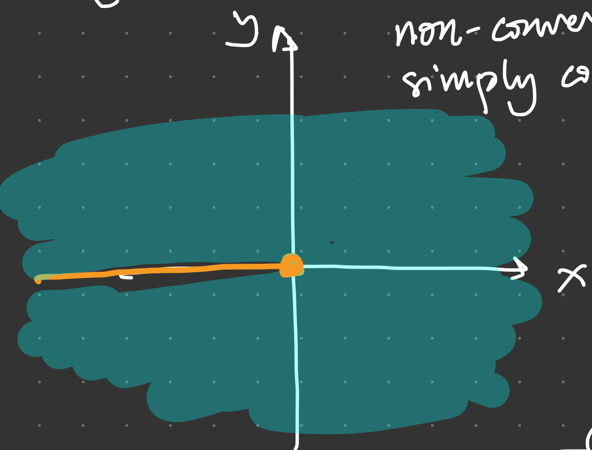
convex and simply-connected



convex and simply-connected

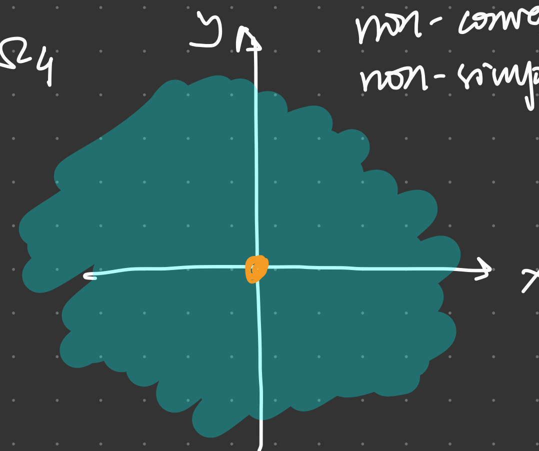
Ω_2

Ω_3



non-convex simply connected.

Ω_4



non-convex non-simply connected

$\Omega_1 \subset \Omega_3 \subset \Omega_4$, $\Omega_2 \subset \Omega_3 \subset \Omega_4$

$F \in C^\infty(\Delta, \mathbb{R}^2)$

$\operatorname{curl} F = 0$?, yes! check that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \iff \operatorname{curl} F = 0.$

For $\Omega_1, \Omega_2, \Omega_3$ F derives from a potential. (Theorem 1)

For $\Omega_4 \longrightarrow$ we don't know.

1) $F = \operatorname{grad} f_1$ in Ω_1. $f_1 = ?$

$$\underset{(1)}{\frac{\partial f_1}{\partial x} = F_1(x,y)}, \quad \underset{(2)}{\frac{\partial f_1}{\partial y} = F_2(x,y)}$$

$$(1) \Rightarrow f_1(x,y) = \int \frac{-y}{x^2+y^2}\,dx + \alpha(y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + \alpha(y)$$

$$f_1(x,y) \Rightarrow (2), \quad \frac{\partial f_1(x,y)}{\partial y} = \cancel{\frac{x}{x^2+y^2}} + \alpha'(y) = F_2 = \cancel{\frac{x}{x^2+y^2}}$$

$\alpha'(y) = 0 \rightarrow \alpha(y) = C, \quad C \in \mathbb{R}.$

$f_1(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + C$. $\operatorname{grad} f_1 = F$ in Ω_1

2) The same analysis as for Ω_1

$$f_2(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + C_2, \quad C_2 \in \mathbb{R}$$

Note: C and C_2 are arbitrary constants.

3) After cases Ω_1 and Ω_2 if a potential exists f_3 s.t. $\operatorname{grad} f_3 = F$ in Ω_3, it must be like:

$$f_3(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + C & \text{if } y > 0 \text{ and } x \in \mathbb{R} \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + C_2 & \text{if } y < 0 \text{ and } x \in \mathbb{R} \end{cases}$$

Then for $x>0$ we have

$$\lim_{\substack{y\to 0^+\\ x>0}} f_3(x,y) = -\lim_{\substack{y\to 0^+\\ x>0}} \operatorname{arctg}\left(\underbrace{\frac{x}{y}}_{+\infty}\right) + C = -\frac{\pi}{2} + C$$

$$\lim_{\substack{y\to 0^-\\ x>0}} f_3(x,y) = -\lim_{\substack{y\to 0^-\\ x>0}} \operatorname{arctg}\left(\underbrace{\frac{x}{y}}_{-\infty}\right) + C_2 = \frac{\pi}{2} + C_2$$

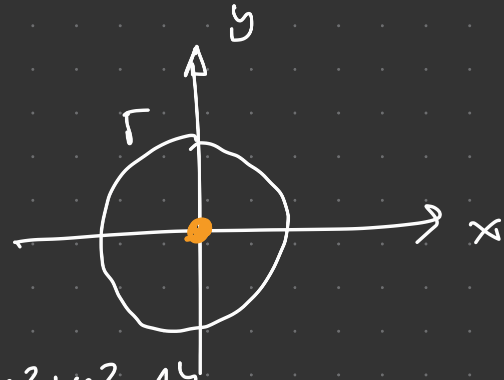
$$\lim_{\substack{y\to 0^+\\ x>0}} f_3 = \lim_{\substack{y\to 0^-\\ x>0}} f_3 \longrightarrow -\frac{\pi}{2} + C = \frac{\pi}{2} + C_2 \longrightarrow C = \pi + C_2$$

$$f_3(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + \pi + C_2 & \text{if } y>0 \\ \pi/2 + C_2 & \text{if } y=0 \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + C_2 & \text{if } y<0 \end{cases}$$

4) Ω_4 is non-convex and non-simply connected.

Theorem 1 says no.

Theorem 2



$$\Gamma = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2=1 \}$$

$$\gamma(t) = (\cos t, \sin t) \ , \ t \in [0, 2\pi].$$

$$\gamma'(t) = (-\sin t, \cos t)$$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) = \int_0^{2\pi} \underbrace{(-\sin t, \cos t)}_{F(\gamma(t))} \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \neq 0$$

$$\int_\Gamma F \cdot dl \neq 0 \rightarrow$$ F does not derive from a potential in Ω_4