19/09/2024

b) $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto F(x,y,z) = (x^2, y^3, z^2)$

and the curve $\Gamma = \{(x,y,z) \in \mathbb{R}^3 : y = e^x$ and $z = x$ for $x \in [0,1]\}$

Parameterization $\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longmapsto \gamma(t) = (t, e^t, t)$

$$\int_\Gamma F \cdot dl = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$\gamma'(t) = (1, e^t, 1)$, $\int_\Gamma F \cdot dl = \int_0^1 (t^2, e^{3t}, t^2) \cdot (1, e^t, 1)\, dt$

$$= \int_0^1 (2t^2 + e^{4t})\, dt = \frac{2}{3} t^3 \Big|_0^1 + \frac{1}{4} e^{4t} \Big|_0^1 = \frac{5 + 3e^4}{4}$$

Example 3: compute the length of Γ, Γ is the circle of radius R and centered at the origin.

$$\Gamma : \{ (x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2 = R^2 \}$$

$$\begin{aligned} \gamma : [0, 2\pi] &\longrightarrow \Gamma \\ t &\longmapsto \gamma(t) = (R\cos t, R\sin t) \end{aligned}$$

$$\int_\Gamma f \, dl = \int_0^{2\pi} f(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

$$\gamma'(t) = (-R\sin t, R\cos t), \quad \|\gamma'(t)\| = R$$

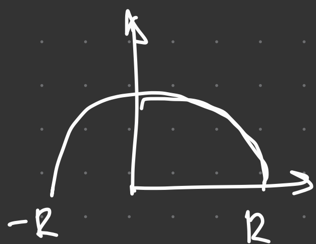
$$\text{length}(\Gamma) = \int_\Gamma 1 \cdot dl$$

f

$f(\gamma(t)) = 1$, $\int_\Gamma f \, dl = \int_0^{2\pi} 1 \cdot \|\gamma'(t)\| \, dt = R \int_0^{2\pi} dt = R\, 2\pi .$

$\gamma(t) = \left(t, \sqrt{R^2 - t^2} \right)$

$t \in [-R, R]$



2.3 Fields that derive from potentials

2.3.1 Description of conservative fields.

- Definition: let $\Omega \subset \mathbb{R}^n$ be an open domain, and

$$F : \Omega \to \mathbb{R}^n$$
$$x \mapsto F(x) = (F_1(x), \dots, F_n(x)) \text{ a vector field}$$

We say that F derives from a potential in Ω if there exists $(\exists)$ a scalar field $f : \Omega \to \mathbb{R}$, $x \mapsto f(x)$ of class $f \in C^1(\Omega)$

s.t. $F = \text{grad } f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$

In this case, F is a <u>conservative field</u> and f is the <u>potential</u>.

Remarks:

1) If the potential exists, it is defined up to a constant $\alpha \in \mathbb{R}$,

$$\operatorname{grad}(f) = \operatorname{grad}(f + \alpha)$$

2) classical example:

$$f = \frac{c}{r} + \alpha$$, r is the distance to a mass M.

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$c = g m M$$

$$F = \frac{-c}{r^3}(x, y, z) \quad , \quad F = \operatorname{grad} f$$

2.3.2 Important results

- Theorem 1: let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$F : \Omega \to \mathbb{R}^n \qquad \text{a vector field s.t. } F \in C^1(\Omega, \mathbb{R}^n)$$
$$x \mapsto F(x)$$

a) Necessary condition: If F derives from a potential in Ω,

then
$$\boxed{\frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}, \ \forall\, i,j = 1, 2, \dots, n \text{ and } \forall\, x \in \Omega} \qquad (*)$$

b) sufficient condition: If $(*)$ holds and if Ω is convex and / or simply connected then F derives from a potential in Ω.

Remarks:

1) The condition (*) is necessary but not sufficient.

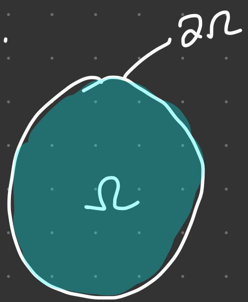
2) The condition (*) is equivalent to curl $F = 0$.

E.g. $n=2$ $\quad \operatorname{curl} F = \dfrac{\partial F_2}{\partial x_1} - \dfrac{\partial F_1}{\partial x_2} = 0$

The same for $n=3$.

Recalls.

1)



Ω is <u>open</u> if $\partial\Omega$ doesn't belong to Ω.

2) $\Omega \subset \mathbb{R}^n$ is <u>convex</u> if $\forall t \in [0,1]$ and $\forall x, y \in \Omega$ we

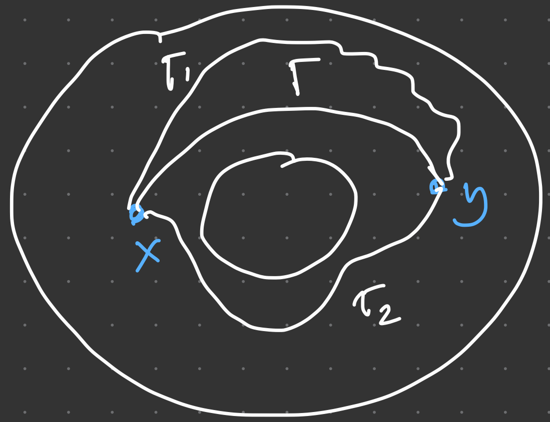
here $x + t(y-x) \in \Omega$ (i.e., the segment between x and y) is fully contained in Ω

convex   non convex.

2) $\Omega \subset \mathbb{R}^n$ is simply connected if $\forall x, y \in \Omega$ $\exists$ a curve Γ than joins x and y that is fully contained in Ω

and being Γ_1 y Γ_2 they can be deformed to Γ.

(i.e, Ω has no holes).



non-connected