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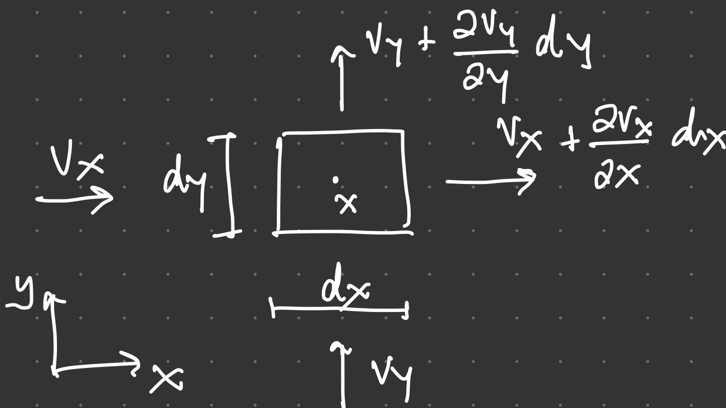
Divergence interpretation



$$dy\left(\cancel{v_x} + \frac{\partial v_x}{\partial x} dx - \cancel{v_x}\right) + \left(\cancel{v_y} + \frac{\partial v_y}{\partial y} dy - \cancel{v_y}\right) dx = \underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)}_{\text{div } V.} dx\,dy .$$

1.4 The curl operator (rotationel)

1.4.1 Definition

Let be $\Omega \subset \mathbb{R}^n$ an open domain. and $F : \Omega \to \mathbb{R}^n$

$$x \longmapsto F(x) = (F_1(x), F_2(x), \dots)$$

a vector field s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

The curl operator (or rotor, or rotationel) denoted curl F (or $\nabla \times F$, or rot F) is defined by

a) When $n=2$, $\operatorname{curl} F(x,y) = \dfrac{\partial F_2}{\partial x}(x,y) - \dfrac{\partial F_1}{\partial y}(x,y)$

b) When $n=3$, $\operatorname{curl} F(x,y,z) =$

$$\left(\frac{\partial F_3}{\partial y}(x,y,z) - \frac{\partial F_2}{\partial z}(x,y,z), \right.$$

$$\frac{\partial F_1}{\partial z}(x,y,z) - \frac{\partial F_3}{\partial x}(x,y,z),$$

$$\left.\frac{\partial F_2}{\partial x}(x,y,z) - \frac{\partial F_1}{\partial y}(x,y,z)\right)$$

$$\nabla \times F = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

1.4.2 Examples:

Example: let us consider the gravitational force of the prev. example with $P_0 = (0,0,0)$

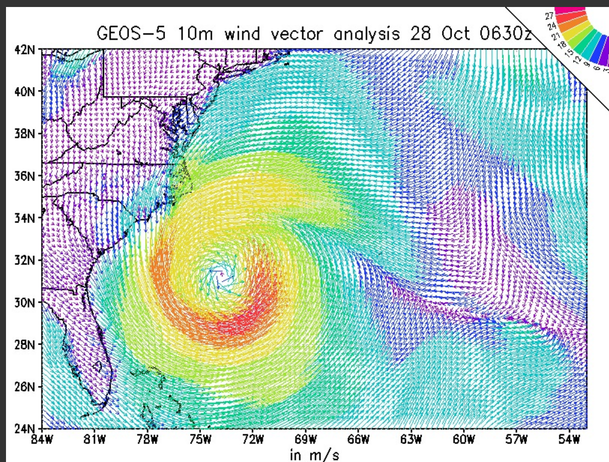
$$F: \mathbb{R}^3 \setminus (0,0,0) \to \mathbb{R}^3$$
$$(x,y,z) \longmapsto F(x,y,z) = -\frac{c}{r^3}(x,y,z)$$

with $r = \sqrt{x^2+y^2+z^2}$ and $c = g\, m M$.

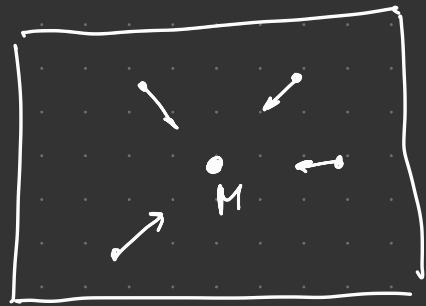
$$\operatorname{curl} F(x,y,z) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{cx}{r^3} & \frac{-cy}{r^3} & \frac{-cz}{r^3} \end{vmatrix} = c \begin{pmatrix} \frac{\partial}{\partial y}\left(\frac{-z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-y}{r^3}\right) \\ \cdots \\ \cdots \end{pmatrix}$$

$$\frac{\partial}{\partial x}\left(-\frac{cy}{r^3}\right) = \frac{d}{dr}\left(\frac{-cy}{r^3}\right)\frac{\partial r}{\partial x}$$

$$= \frac{3c}{r^5} \begin{pmatrix} zy - yz \\ xz - zx \\ yx - xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



gravitational field



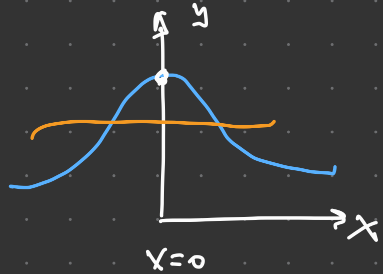
1.5 The laplacian operator

let $\Omega \subset \mathbb{R}^n$ an open domain and $f : \Omega \to \mathbb{R}$
$x \mapsto f(x)$

a scalar field s.t. $f \in C^2(\Omega)$. The laplacian of f, denoted as Δf (or $\nabla^2 f$), is defined by:

$$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \dots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

1.5.2 Examples:



Example:

$$f : \mathbb{R}^3 \setminus (0,0,0) \longrightarrow \mathbb{R}$$

$$(x,y,z) \longmapsto f(x,y,z) = \frac{gmM}{r}$$

$$\frac{\partial f}{\partial x} = \frac{d\left(\frac{gmM}{r}\right)}{dr}\frac{\partial r}{\partial x} \quad , \quad \frac{\partial^2 f}{\partial x^2} = \frac{d^2\left(\frac{gmM}{r}\right)}{dr^2}\frac{\partial r}{\partial x} + \dots$$

$$= -c\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -3c\left(\frac{1}{r^3} - \frac{r^2}{r^5}\right) = 0.$$

1.6 Differentiation formulas

1.6.1 Important results

• Theorem : let $\Omega \subset \mathbb{R}^n$ an open domain, let $f : \Omega \to \mathbb{R}$

s.t. $f \in C^2(\Omega)$, and let $F : \Omega \to \mathbb{R}^n$ s.t. $F \in C^2(\Omega)$

then:

1) $\operatorname{div}(\operatorname{grad} f) = \Delta f$,

2) if $n=2$ we have $\operatorname{curl}(\operatorname{grad} f) = 0 \quad (\mathbb{R})$

3) if $n=3$ we have $\operatorname{curl}(\operatorname{grad} f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\mathbb{R}^3)$

4) if $n=3$ we have $\operatorname{div}(\operatorname{curl} F) = 0$