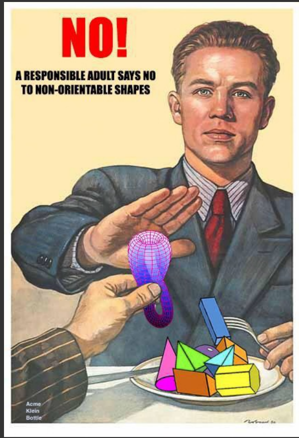
10/10/2024

- Definition 3:

A (piecewise) regular surface is <u>orientable</u> if $\exists$ a field of normals $\nu : \Sigma \to \mathbb{R}^3$ <u>continuous</u>.

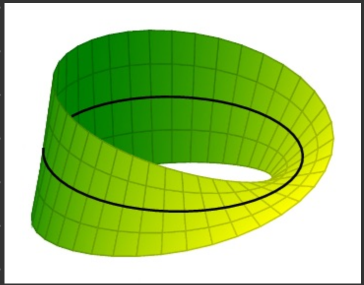
Such field of normal is called a <u>orientation</u> of Σ



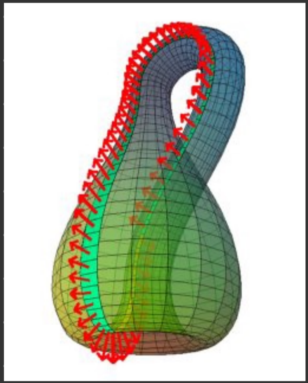
Remark: the vast majority of surfaces are orientable

But $\exists$ counter-examples

Moebius strip.



Klein bottle



3.2 Surface integrals

3.2.1 Scalar fields integrals

- Definition: Let $\Sigma \subset \mathbb{R}^3$ be a regular surface parameterized by

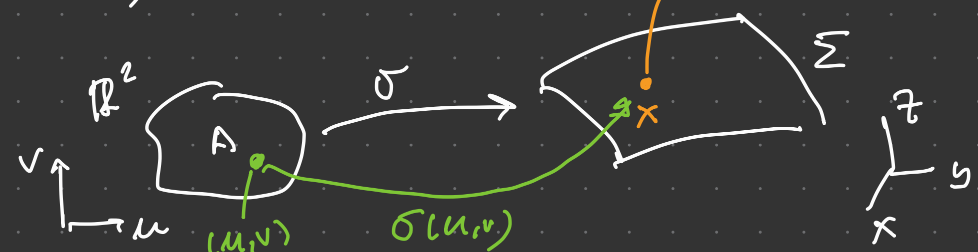
$$\sigma : \bar{A} \longrightarrow \Sigma, \quad (u,v) \longmapsto \sigma(u,v)$$

and let be $f : \Sigma \to \mathbb{R}, \; x \longmapsto f(x)$ be

a continuous field $(f \in C^0(\Sigma))$

The integral of f over Σ is defined by

$$\iint_\Sigma f \, dS = \iint_A f(\sigma(u,v)) \, \| \sigma_u \times \sigma_v \| \, du \, dv$$



Remarks

a) Analogy with curvilinear integral of a scalar field along Γ

$$\gamma : [a,b] \longrightarrow \Gamma \qquad t \longmapsto \gamma(t) \quad ; \quad g : \Gamma \longrightarrow \mathbb{R} \qquad x \longmapsto g(x)$$

$$\sigma : A \longrightarrow \Sigma \qquad (u,v) \longmapsto \sigma(u,v) \quad ; \quad f : \Sigma \longrightarrow \mathbb{R} \qquad x \longmapsto f(x)$$

b) Compute area of Σ (using $f=1$)

$$\text{area}(\Sigma) = \iint_\Sigma 1 \, ds = \iint_A 1 \cdot \| \sigma_u \times \sigma_v \| \, du \, dv$$

$$\text{length}(\Gamma) = \int_\Gamma 1 \, dl = \int_a^b 1 \, \| \gamma'(t) \| \, dt$$

c) For a piecewise regular surface $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ we define

$$\iint_{\Sigma} f \, ds = \sum_{i=1}^{k} \iint_{\Sigma_i} f \, ds$$

3.2.2 Examples:

- Example 1: Let be $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2 \}$

Compute area of Σ

$$\text{area}(\Sigma) = \iint_{\Sigma} 1 \cdot ds = \iint_{A} 1 \, \|\sigma_u \times \sigma_v\| \, du\,dv$$

$$\sigma : \bar{A} \longrightarrow \Sigma$$

$$(\theta, \varphi) \longmapsto \sigma(\theta,\varphi) = (R\sin\varphi\cos\theta, \, R\sin\varphi\sin\theta, \, R\cos\varphi)$$

$\sigma_u \times \sigma_v = -R\sin\varphi \, \sigma(\theta,\varphi)$, $\|\sigma_u \times \sigma_v\| = R\sin\varphi \, \|\sigma(\theta,\varphi)\|$

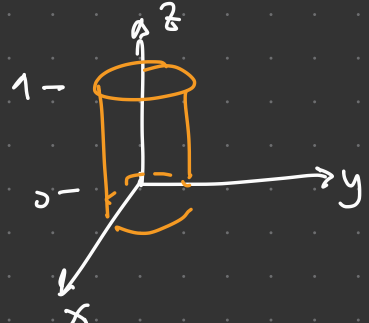
$\varphi \in [0,\pi] \longrightarrow$

$$= R^2 \sin\varphi$$

$$\text{area}(\Sigma) = \int_0^{2\pi} \left(\int_0^{\pi} R^2 \sin\varphi \, d\varphi \right) d\theta = 2\pi R^2 \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= 4\pi R^2$$

Example 2: let $\Sigma = \{ (x,y,z) \in \mathbb{R}^2 : x^2+y^2=1 \text{ and } 0 \leq z \leq 1 \}$

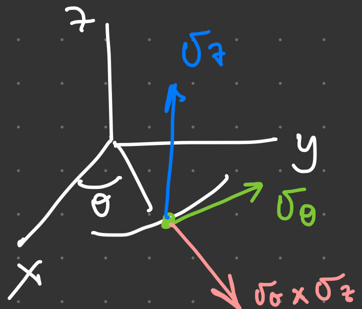


$$f : \Sigma \to \mathbb{R}$$
$$(x,y,z) \mapsto f(x,y,z) = x^2+y^2+2z$$

$$\iint_\Sigma f \, ds = \iint_A f(\sigma(u,v)) \, \|\sigma_u \times \sigma_v\| \, du\,dv$$

$$\sigma(\theta, z) = (\cos\theta, \sin\theta, z), \quad A =]0, 2\pi[\times]0, 1[$$

$\|\sigma_\theta \times \sigma_z\| = 1$,, $\quad \sigma_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

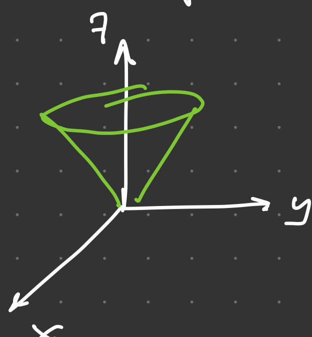


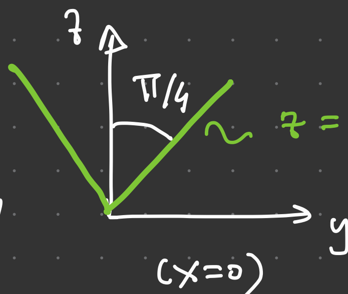
$$f(\sigma(\theta, z)) = 1 + 2z$$

$$\iint_\Sigma f\, ds = \int_0^{2\pi} d\theta \int_0^1 (1 + 2z)\, dz = 2\pi \int_0^1 (1+2z)\, dz$$

$$= 4\pi .$$

- Example 3: let $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } 0 \le z \le 1 \}$





$\sim z = |y| \quad (z^2 = y^2)$

$$f(x,y,z) = xy + z^2$$

$$\sigma(\theta, z) = (z\cos\theta,\ z\sin\theta,\ z)$$

$$\sigma_\theta \times \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix} \qquad \| \sigma_\theta \times \sigma_z \| = \sqrt{2}\, z$$

$$f(\sigma(\theta, z)) = z^2 \sin\theta\cos\theta + z^2$$

$$\iint_\Sigma f\, dS = \int_0^{2\pi}\int_0^1 (z^2 \sin\theta\cos\theta + z^2)\sqrt{2}\, z\, dz\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} (\sin\theta\cos\theta + 1)\, d\theta \int_0^1 z^3\, dz = \frac{\sqrt{2}}{4}\left(\frac{1}{2}\sin^2\theta + \theta\right)\Big|_0^{2\pi}$$

mistake during lecture.

$$= \frac{\sqrt{2}}{2}\pi$$