03/10/2024

2.4.5 Corollaries of Green's theorem

- Corollary 1: Divergence theorem in the plane.

Let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $\nu : \partial A \to \mathbb{R}^2$ the field of outer unit normal vectors defined as $\nu = (\nu_1, \nu_2)$.

Let $F : \bar{A} \to \mathbb{R}^2$ be a vector field s.t. $F \in C^1(\bar{A}, \mathbb{R}^2)$ defined as $F = (F_1, F_2)$. Then:

$$\iint_A \operatorname{div} F(x,y)\, dx\, dy = \int_{\partial A} F \cdot \nu \, dl$$

In other words:

$$\iint_A \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx\,dy = \int_{\partial A} (F_1 \nu_1 + F_2 \nu_2)\, dl$$

- Corollary 2:

Let $A \subset \mathbb{R}^2$ a regular domain s.t. ∂A is positively oriented

Let's consider the vector fields F, G_1, and $G_2 : \bar{A} \to \mathbb{R}^2$

defined $F(x,y) = (-y, x)$, $G_1(x,y) = (0, x)$,

$G_2(x,y) = (-y, 0)$

Then $\text{Area}(A) = \frac{1}{2} \int_{\partial A} F \cdot dl = \int_{\partial A} G_1 \cdot dl = \int_{\partial A} G_2 \cdot dl$

- Corollary 3:

Let $A \subset \mathbb{R}^2$ be a regular domain whose boundary ∂A is positively oriented. Let $\nu : \partial A \to \mathbb{R}^2$ the field of outer unit normal and let $f : \bar{A} \to \mathbb{R}$ be a scalar field s.t. $f \in C^2(\bar{A})$. Then:

$$\iint_A (\Delta f)(x,y)\,dx\,dy = \int_{\partial A} \operatorname{grad} f \cdot \nu \, dl$$

Proof:

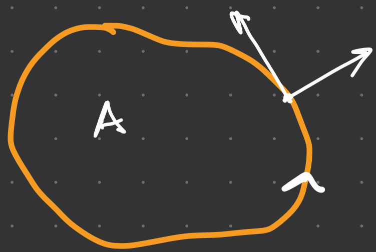
$$\iint_A (\Delta f)(x,y)\,dx\,dy \overset{\Delta f = \operatorname{div}(\operatorname{grad} f)}{=} \iint_A \operatorname{div}(\underbrace{\operatorname{grad} f}_{F})(x,y)\,dx\,dy$$

$$= \iint_A \operatorname{div} F(x,y)\,dx\,dy \overset{\text{Div. thm}}{=} \int_{\partial A} F(x,y)\cdot \nu(x,y)\,dl = \int_{\partial A} \operatorname{grad} f \cdot \nu\, dl \qquad \square$$

Proof of Divergence theorem:

$\|T\| = 1$

$\hookrightarrow \|\nu\| = 1$



$T = (T_1, T_2)$

$\nu = (\nu_1, \nu_2) = (T_2, -T_1)$

If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $t \in [a,b]$ is a parameterization of ∂A, then we have

$$T = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{(\gamma_1'(t), \gamma_2'(t))}{\|\gamma'(t)\|} = (T_1, T_2)$$

$\nu = (T_2, -T_1)$ is the outer unit normal vector (pointing out of A).

$F = (F_1, F_2)$.

$\phi = (-F_2, F_1)$ thus:

$$\operatorname{curl} \phi = \frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial y} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \operatorname{div} F$$

Then:

$$\iint_A \operatorname{div} F(x,y)\,dx\,dy = \iint_A \operatorname{curl} \phi\,(x,y)\,dx\,dy \overset{\text{Green's thm.}}{=} \int_{\partial A} \phi \cdot d\ell$$

$$= \int_a^b \phi(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b \phi(\gamma(t)) \cdot \underbrace{\frac{\gamma'(t)}{\|\gamma'(t)\|}}_{T} \|\gamma'(t)\|\,dt$$

$$= \int_a^b \phi(\gamma(t)) \cdot T(t)\, \|\gamma'(t)\|\,dt$$

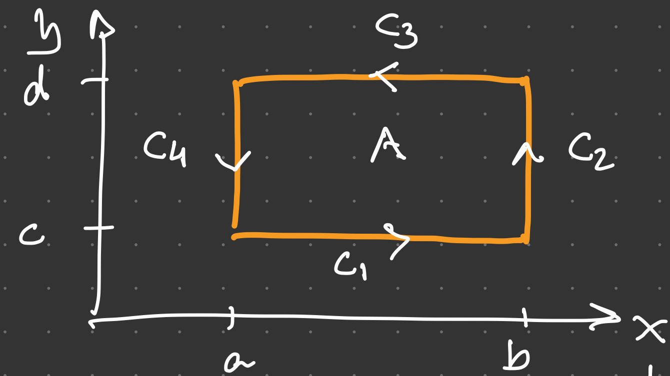
$$= \int_a^b \Big(\underbrace{\phi_1(\gamma(t))}_{-F_2(\gamma(t))} \underbrace{T_1(t)}_{= -\nu_2(t)} + \underbrace{\phi_2(\gamma(t))}_{F_1(\gamma(t))} \underbrace{T_2(t)}_{= \nu_1(t)} \Big) \|\gamma'(t)\|\,dt$$

$$= \int_a^b \underbrace{\left(F_2(\gamma(t))\, \nu_2(t) + F_1(\gamma(t))\, \nu_1(t) \right)}_{F(\gamma(t)) \cdot \nu(t)} \|\gamma'(t)\|\, dt$$

$$= \int_a^b (F \cdot \nu)(\gamma(t))\, \|\gamma'(t)\|\, dt = \int_{\partial A} F \cdot \nu\, d\ell \qquad \square$$

2.4.4 Green's theorem proof

For the sake of simplicity, we prove it for the case that A is a rectangle : $A = (a,b) \times (c,d)$



let $F : \bar{A} \to \mathbb{R}^2$ be a vector field

$$F(x,y) = (F_1(x,y), F_2(x,y))$$

$$F \in C^1(A, \mathbb{R}^2)$$

- $$\iint_A \operatorname{curl} F(x,y)\,dx\,dy = \int_c^d \int_a^b \left(\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) \right) dx\,dy$$

$$= \int_c^d \left(\int_a^b \frac{\partial F_2}{\partial x}\,dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial F_1}{\partial y}\,dy \right) dx$$

$$= \int_c^d \left(F_2(b,y) - F_2(a,y) \right) dy - \int_a^b \left(F_1(x,d) - F_1(x,c) \right) dx$$

- $$\int_{\partial A} F \cdot dl = \int_{C_1} F \cdot dl + \int_{C_2} F \cdot dl + \int_{C_3} F \cdot dl + \int_{C_4} F \cdot dl$$

$$C_1 = \{ \gamma_1(t) = (t, c) \in \mathbb{R}^2 : t \in [a,b] \}$$

$$C_2 = \{ \gamma_2(t) = (b, t) \in \mathbb{R}^2 : t \in [c,d] \}$$

$$C_3 = \{ \gamma(t) = (+t, d) \in \mathbb{R}^2 : t \in [a,b] \}$$

$$C_4 = \{ \gamma(t) = (a, +t) \in \mathbb{R}^2 : t \in [c,d] \}$$

Note: the green - signs in C_3 and C_4 where a mistake, and I replaced them by positive signs $(+)$.

C_3 and C_4 do not follow the positive orientation of the boundary.

A possible way of overcomming this is to just change the curve of orientation by permuting the integration limits as:

$$\int_{C_3} F \cdot dl = \int_{b}^{a} F(\gamma_3(t)) \cdot \gamma_3'(t)\, dt.$$ The same applies to C_4.