EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Exercises**
Series 14	December 17, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1 (Ex 18.1 page 304)**.**
Solve for $u = u(x,t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \cos x - \cos(3x), & x \in (0,\pi). \end{cases}$$

Exercise 2 (Ex 18.3 page 306)**.**
Solve for $u = u(x,t)$

$$\begin{cases} (2+t)\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin x + 4\cos(2x), & x \in (0,\pi). \end{cases}$$

Hint: A non-trivial solution $w(x)$ of the Ordinary Differential Equation (ODE):

$$(a+bx)w'(x) + cw(x) = 0\,,$$

with $a, b, c \in \mathbb{R}$, such that $b \neq 0$ and $c \neq 0$, is:

$$w(x) = (a+bx)^{-c}\,.$$

Exercise 3 (Ex 18.4 page 307)**.**
Solve

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & x \in (0,1),\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, & t > 0 \\ u(x,0) = 0, & x \in (0,1) \\ \dfrac{\partial u}{\partial t}(x,0) = 1 + \cos^2(\pi x), & x \in (0,1). \end{cases}$$

Exercise 4 (Ex 18.5 page 308)**.**
Find a solution $u = u(x,t)$ of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} = 0, & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ \dfrac{\partial^2 u}{\partial x^2}(0,t) = \dfrac{\partial^2 u}{\partial x^2}(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin x + \sin(3x), & x \in (0,\pi). \end{cases}$$

Exercise 5 (Ex 18.6 page 309)**.**
Find a solution $u = u(x,t)$ of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial x} + 2u, & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = e^{-x}\sin(2x), & x \in (0,\pi). \end{cases}$$

Hint: A non-trivial solution $v(t)$ of the ODE (damped oscillator):

$$mv''(t) + cv'(t) + kv(t) = 0\,,$$

with $m, c, k \in \mathbb{R}^+$, is:

$$v(t) = e^{-\frac{c}{2m}t}\sin(\omega t)\,,$$

with $\omega = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$.