EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Exercises**
Series 10	November 19, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Hint: For the following exercises, we suggest to:

1. *Start by sketching the graphs of f and f', over at least two periods;*
2. *Check that the function f is (piecewise) C^1;*

Exercise 1 (Ex 14.1 page 219)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $f(x) = e^{(x-\pi)}$ over $[0, 2\pi[$.

1. Sketch the graph of f and the graph of f'.
2. Calculate the Fourier series Ff of the function f.
3. With the help of the Dirichlet theorem, compare Ff and f over $[0, 2\pi]$.
4. With the help of the two previous questions, show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{e^{\pi} - e^{-\pi}}.$$

Exercise 2 (Ex 14.2 page 220)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $f(x) = (x-\pi)^2$ over $[0, 2\pi[$.

1. Sketch the graph of f and the graph of f'.
2. Calculate the Fourier series Ff of the function f.
3. With the help of the Dirichlet theorem, compare Ff and f over $[0, 2\pi]$.
4. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 3 (Ex 14.4 page 220)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by periodicity of period 2 such that

$$f(x) = x \quad \text{if } x \in [0, 2[.$$

Calculate the complex Fourier series.

Exercise 4 (Ex 14.11 page 221).

1. Using complex notations, calculate the Fourier series of the 2π-periodic and odd function defined on $[0, \pi]$ by

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq \frac{\pi}{2}, \\ \pi - x & \text{if} \quad \frac{\pi}{2} < x \leq \pi. \end{cases}$$

2. Then deduce

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}.$$

Exercise 5 (Ex 14.3 page 220).
Calculate the Fourier series of the function $f : \mathbb{R} \to \mathbb{R}$ 2π-periodic defined by

$$f(x) = \begin{cases} \sin(x) & \text{if} \quad 0 \leq x \leq \frac{\pi}{2}, \\ 0 & \text{if} \quad \frac{\pi}{2} < x < \frac{3\pi}{2}, \\ \sin(x) & \text{if} \quad \frac{3\pi}{2} \leq x < 2\pi. \end{cases}$$

Exercise 6 (Ex 14.8 page 221).

1. Calculate the Fourier series of the 2π-periodic function defined by

$$f(x) = |\cos(x)| \quad \text{if} \quad x \in [0, 2\pi[.$$

2. Deduce the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}.$$

Exercise 7 (Ex 14.10 page 221).

1. For $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, calculate the Fourier series of the 2π-periodic function defined by

$$f(x) = \cos(\alpha x) \quad \text{if} \quad x \in [-\pi, \pi[.$$

2. Deduce the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)}.$$