

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Aide-memoire: To verify the divergence theorem in $\mathbb{R}^3$, proceed as follows:

1. Sketch the domain Ω , then calculate $\operatorname{div} F(x, y, z)$.
2. Parameterize the domain Ω . Use this parameterization to express

$$\iiint_{\Omega} \operatorname{div} F(x, y, z) \, dx \, dy \, dz$$

as a triple integral where the bounds and the function to be integrated are explicitly indicated.

3. Write $\partial\Omega$ as a union of regular surfaces; for each, give a parameterization and a field of exterior normals. Add this to your sketch.
4. Express

$$\iint_{\partial\Omega} F \cdot \nu \, ds$$

as a sum of double integrals where the bounds and the functions to be integrated are explicitly indicated.

5. Check the conclusion of the divergence theorem for Ω and F .

Exercise 1 (Ex 6.4 page 66).

Verify the divergence theorem for $F(x, y, z) = (x^2, y^2, z^2)$ and $\Omega = \{(x, y, z) \in \mathbb{R}^3 : b^2(x^2 + y^2) < a^2 z^2 \text{ and } 0 < z < b\}$.

Exercise 2 (Ex 6.3 page 66).

Verify the divergence theorem for $F(x, y, z) = (xy, yz, xz)$ and $\Omega = \{(x, y, z) \in \mathbb{R}^3 : 0 < z < 1 - x - y, 0 < y < 1 - x, 0 < x < 1\}$.

Exercise 3 (Ex 6.6 page 66).

Verify the divergence theorem for $F(x, y, z) = (x, y, z)$ and $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } x^2 + y^2 < 3z\}$.

Exercise 4 (Ex 6.9 page 67).

Verify the divergence theorem for $F(x, y, z) = (2, 0, xy^2 + z^2)$ and $\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : x > 0, 0 < z < 2 \text{ and } 4(x^2 + y^2) < (z - 4)^2 \right\}$.

Exercise 5 (Ex 6.11 page 67).

Let $\Omega \subset \mathbb{R}^3$ a regular domain and ν a field of unit normals exterior to Ω . Let the vector fields F , G_1 , G_2 , and G_3 defined by

$$\begin{aligned} F(x, y, z) &= (x, y, z), & G_1(x, y, z) &= (x, 0, 0), \\ G_2(x, y, z) &= (0, y, 0), & G_3(x, y, z) &= (0, 0, z). \end{aligned}$$

Prove that:

1. $\text{Volume}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu) \, ds$
2. $\text{Volume}(\Omega) = \iint_{\partial\Omega} (G_i \cdot \nu) \, ds$ for $i = 1, 2, 3$.