EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Exercises**
Series 6	October 15, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Reminder : In order to compute an integral over a surface (like in Exercise 2 and parts 4 and 5 of Exercise 3), proceed as follows:

1. *Sketch the surface* Σ.
2. *Provide a parameterization* $\sigma : \bar{A} \to \Sigma$ *of the surface* Σ. *Then, define the parametric domain* $\bar{A}$ *and the function* σ.
3. *Provide a normal vector and add it to your sketch.*
4. *Use this parameterization to express the integral as a multiple integral where the bounds and the function that need to integrated are indicated explicitly.*
5. *Compute the integral.*

Exercise 1 (Ex 4.8 page 43 and Ex 4.7 page 42)**.**
Let $\Omega \subset \mathbb{R}^2$ be a regular set whose border $\partial\Omega$ is oriented positively. Let ν be a field of external unit normals to $\partial\Omega$. Let F be a vector field such that $F \in C^1\left(\bar{\Omega}, \mathbb{R}^2\right)$ and f a scalar field such that $f \in C^2\left(\bar{\Omega}\right)$. Prove that:

1. $\displaystyle\iint_{\Omega} \operatorname{div} F\,(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} (F \cdot \nu)\,\mathrm{d}l$

   Indication: Write $F = (F_1, F_2)$ and apply Green's theorem to the vector field $\Phi = (-F_2, F_1)$.

2. $\displaystyle\iint_{\Omega} \Delta f\,(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} (\operatorname{grad} f \cdot \nu)\,\mathrm{d}l.$

Exercise 2 (Ex 5.1 page 56)**.**
Let $f(x,y,z) = xy + z^2$ and

$$\Sigma = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\right\}.$$

Compute $\displaystyle\iint_{\Sigma} f\mathrm{d}s$.

Exercise 3 (Ex 5.6 page 57)**.**
Let $0 < a < R$. In $\mathbb{R}^3$ we consider the torus Ω obtained with the rotation of the disk $(x-R)^2 + z^2 \leq a^2$ around the Oz axis and its parameterization is:

$$x = (R + r\cos\varphi)\cos\theta, \qquad y = (R + r\cos\varphi)\sin\theta, \qquad z = r\sin\varphi,$$

with $0 < r < a$, $0 \leq \theta < 2\pi$, $0 \leq \varphi < 2\pi$.

1. Sketch Ω and indicate what does r, θ, and φ represent.
2. Compute the Jacobian of the transformation that describes Ω in terms of the variables r, θ, and φ.
3. Compute the volume of Ω.
4. Write a regular parameterization of the surface of the torus (noted $\partial\Omega$) and compute a normal to $\partial\Omega$.
5. Compute the area of $\partial\Omega$.
6. Compute $\displaystyle\iiint_\Omega z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

The exercises below are a recap of Analysis II

Exercise 4.

1. Let $D = \big\{(x,y) \in \mathbb{R}^2 : \, x \geq 0, \, y \geq 0 \text{ and } x + y \leq 1\big\}$.
   Compute $\displaystyle\iint_D \sqrt{1-x-y}\, \mathrm{d}x\mathrm{d}y$.
2. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2(x + \sqrt{x^2+y^2})\}$.
   Compute $\displaystyle\iint_D \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{3}{4}}}$.
3. Let $D = \big\{(x,y,z) \in \mathbb{R}^3 : \, x \geq 0, \, y \geq 0, \, z \geq 0, \; z \leq 1 - y^2 \text{ and } x + y \leq 1\big\}$.
   Compute $\displaystyle\iiint_D z \, \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Exercise 5.
Compute the volume of:

1. $D = \big\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \text{ and } x^2 + y^2 \leq z\big\}$.
2. $D = \{(x,y,z) \in \mathbb{R}^3 : x+y+z \leq \sqrt{2}, \; x^2+y^2 \leq 1, \; x \geq 0, \; y \geq 0 \text{ and } z \geq 0\}$.