EPFL - Fall 2024	Dr. Pablo Antolin
Analysis III - 202 (c) GM EL MX	**Exercises**
Series 5	October 8, 2024

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Hint : In order to verify Green's theorem (e.g., exercises 1 and 2), apply the following steps:

1. Try to sketch the domain Ω and its boundary $\partial\Omega$. Indicate the direction of the curve $\partial\Omega$ so that it is positively oriented.
2. Compute $\mathrm{curl} F(x, y)$.
3. Parameterize the domain Ω, and use this parameterization to compute
$$\iint_\Omega \mathrm{curl} F \, \mathrm{d}x\mathrm{d}y.$$
4. Parametrize the boundary $\partial\Omega$ of Ω, and use this parameterization to compute
$$\int_{\partial\Omega} F \cdot \mathrm{d}l.$$
5. To verify the Green's theorem, check that both integrals are equal.

Exercise 1 (Ex 4.1 and Ex 4.2 page 41)**.**
Verify Green's theorem in the following cases:

1. $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$ and $F(x, y) = (xy, y^2)$.
2. $A = \left\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\right\}$ and $F(x, y) = (x + y, y^2)$.

Exercise 2 (Ex 4.4i and Ex 4.5 page 42)**.**
Verify Green's theorem in the following cases:

1. $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 1\right\}$ and $F(x, y) = (-x^2 y, xy^2)$.
2. $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x^2 - 4 < y < 2\right\}$ and $F(x, y) = (xy, y)$.

Exercise 3 (Ex 4.3 page 42)**.**
Let $\Omega \subset \mathbb{R}^2$ be a triangle whose vertices are $(0,0)$, $(0,1)$, and $(1,0)$. Let be $f(x,y) = y + e^x$. Compute:

1. $\displaystyle\int_\Omega \Delta f(x,y)\,\mathrm{d}x\mathrm{d}y$.
2. $\displaystyle\int_{\partial\Omega} \left(\frac{\partial f}{\partial x}\nu_1 + \frac{\partial f}{\partial y}\nu_2\right) \mathrm{d}l$, where $\nu = (\nu_1, \nu_2)$ is the outer unit vector normal of the boundary $\partial\Omega$.

Exercise 4 (Ex 4.9i page 43)**.**
Let $\Omega \subset \mathbb{R}^2$ be a regular domain which has a positively oriented boundary $\partial\Omega$. Let F, G_1, and G_2 be the vector fields defined as

$$F(x,y) = (-y, x), \quad G_1(x,y) = (0, x) \quad \text{and} \quad G_2(x,y) = (-y, 0).$$

Prove that:

1. $\displaystyle\text{Area}(\Omega) = \frac{1}{2}\int_{\partial\Omega} F \cdot \mathrm{d}l$.
2. $\displaystyle\text{Area}(\Omega) = \int_{\partial\Omega} G_1 \cdot \mathrm{d}l.$.
3. $\displaystyle\text{Area}(\Omega) = \int_{\partial\Omega} G_2 \cdot \mathrm{d}l$.