

Note: several exercises are extracted from [B.Dacorogna and C.Tanteri, *Analyse avancée pour ingénieurs* (2018)]. Their corrections can be found there.

Exercise 1 (Ex 1.1 page 7).

Let

$$F(x, y, z) = (y^2 \sin(xz), e^y \cos(x^2 + z), \ln(2 + \cos(xy))) = (F_1, F_2, F_3).$$

Compute:

1. $\text{grad } F_1, \text{grad } F_2, \text{grad } F_3$
2. $\text{div } F$
3. $\text{rot } F$.

Exercise 2 (Ex 1.2 page 7).

Which of the following expressions are correct in the case that $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is of class $C^1(\mathbb{R}^3)$ and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is of class $C^1(\mathbb{R}^3; \mathbb{R}^3)$?

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|--------------------------|------------------------|-----------------------------|-----------------------|
| a) $\text{grad } f$ | b) $f \text{ grad } f$ | c) $F \cdot \text{grad } f$ | d) $\text{div } f$ |
| e) $\text{div}(fF)$ | f) $\text{rot}(fF)$ | g) $\text{rot } f$ | h) $f \text{ rot } F$ |
| i) $\text{rot div } F$. | | | |

Exercise 3 (Ex 1.3 page 5).

Let $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$, and r such that $r = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}$. Let f be a scalar field defined by $f(x) = 1/r$. Compute Δf .

Exercise 4 (Ex 1.6 and 1.7 page 8).

Let $\Omega \subset \mathbb{R}^3$ be an open domain. Verify that:

1. If $f \in C^1(\Omega)$ and $g \in C^2(\Omega)$, then:

$$\text{div}(f \text{ grad } g) = f \Delta g + \text{grad } f \cdot \text{grad } g$$

2. If $f, g \in C^1(\Omega)$, then:

$$\text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$$

3. If $f \in C^1(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^3)$ then:

$$\operatorname{div}(fF) = f \operatorname{div} F + F \cdot \operatorname{grad} f$$

4. If $F \in C^2(\Omega, \mathbb{R}^3)$, then:

$$\operatorname{rot} \operatorname{rot} F = -\Delta F + \operatorname{grad} \operatorname{div} F,$$

where $\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)$ for a given vector field $F = (F_1, F_2, F_3)$.

5. If $f \in C^1(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^3)$, then:

$$\operatorname{rot}(fF) = \operatorname{grad} f \times F + f \operatorname{rot} F$$

Exercise 5 (Ex 1.4 page 7).

Let $f \in C^2(\Omega)$, where:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}.$$

1. In the case where we have:

$$g(r, \theta) := f(r \cos \theta, r \sin \theta) = f(x, y),$$

show that:

$$\frac{\partial^2 g(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial g(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 g(r, \theta)}{\partial \theta^2} = \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = \Delta f(x, y).$$

2. Compute Δf when:

$$f(x, y) := \sqrt{x^2 + y^2} + \left(\arctan \frac{y}{x} \right)^2.$$