Correction of Examen of Analysis III 18.01.2022

1 QCM

Question 1.
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by:

$$f(x, y) = x^2 + y^3 + xy$$

Its Laplacian $\Delta f = \operatorname{div}(\nabla f)$ at (x, y) is

Correct answer: $2 + 6y$

Question 2.
Consider a curve $C \subset \mathbb{R}^2$ connecting the point $P = (1, 0)$ to the point $Q = (0, 1)$, and let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$G(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2} \right)$$

Then:

Correct answer: $\displaystyle\int_C G \cdot \mathrm{dr} = 0$

Question 3.
The non-zero complex Fourier coefficients of the function g defined by:

$$g(x) = \cos(x) + 3\sin(3x),$$

which enable to express g as:

$$g(x) = \sum_{z \in \mathbb{Z}} c_k \, e^{ikx},$$

are:

Correct answer: c_1, c_{-1}, c_3, c_{-3}

Question 4.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$F(x, y) = (x, y),$$

and let Γ be the curve defined by:

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\},$$

with $R \in \mathbb{R}, R > 0$. The line integral $\displaystyle\int_\Gamma F \cdot \mathrm{d}l$ is equal to:

Correct answer: 0

Question 5.

Let $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, with $\Omega \subset \mathbb{R}^2$, be the vector field defined by:

$$F(x, y) = \left(x, 4\frac{x}{y}\right).$$

Does the field F derive from a potential?

Correct answer: No.

2 Open questions

Question 6.

1. We have that $\gamma'(t) = (\cos(2t), \cos(t))$. By direct calculation we obtain:

$$\begin{aligned}
\int_C \phi \,\mathrm{d}l &= \int_0^\pi \phi(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t \\
&= \int_0^\pi \begin{pmatrix} -\sin(t) \\ \frac{1}{2}\sin(2t) \end{pmatrix} \cdot \begin{pmatrix} \cos(2t) \\ \cos(t) \end{pmatrix} \mathrm{d}t \\
&= \int_0^\pi \left(-\sin(t)\cos(2t) + \frac{1}{2}\sin(2t)\cos(t) \right) \mathrm{d}t \\
&= \int_0^\pi \left(-\sin(t)[\cos^2(t) - \sin^2(t)] + \sin(t)\cos^2(t) \right) \mathrm{d}t \\
&= \int_0^\pi \sin^3(t)\mathrm{d}t \\
&= \int_0^\pi \sin(t)\left(1 - \cos^2(t)\right) \mathrm{d}t \\
&= 2 - \int_0^\pi \sin(t)\cos^2(t)\mathrm{d}t
\end{aligned}$$

Integrating by parts with $u = \cos(t)$ and $\mathrm{d}u = -\sin(t)\mathrm{d}t$:

$$\begin{aligned}
&= 2 - \int_{-1}^1 u^2 \mathrm{d}u = 2 - \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}$$

2. We have curl $\phi = \partial_x \phi_2 - \partial_y \phi_1 = 2$. Applying Green's theorem:

$$\int_C \phi \,\mathrm{d}l = \int_R \left(\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial y} \right) dx\mathrm{d}y = 2 \int_R \mathrm{d}x\mathrm{d}y = 2 \cdot \mathrm{Area}(R).$$

The curvilinear integral computed in (1) is the double of the area of the surface R.

Common mistakes: part 1

The most common mistakes arise from the integration of sine/cosine (missing signs) and computation. Most mistakes were on the integration of $\sin(t)\cos^2(t)$ (sign error the most common one).

Common mistakes: part 2

Citing Green's theorem without computing curl$(F) = 2$ it is not sufficient to get full points, the important observation was that curl$(F) = 2$, then it is constant and give a precise relation between the area of R and the curvilinear integral.

Question 7.

1. We remark that $\mathbb{R}^3$ is convex and simply connected. Under this hypothesis, a necessary and sufficient condition for F deriving from a potential is given by $\mathrm{curl}F = 0$ for every $(x,y,z) \in \mathbb{R}^3$. A direct calculation gives us:

$$\mathrm{curl}F = \mathrm{curl}\begin{pmatrix} x^2+5\lambda y+3yz \\ 5x+3\lambda xz-2 \\ (2+\lambda)xy-4z \end{pmatrix} = \begin{pmatrix} (2+\lambda)x-3\lambda x \\ 3y-(2+\lambda)y \\ 5+3\lambda z-5\lambda-3z \end{pmatrix}.$$

We obtain the following conditions, that must be true for every point $(x,y,z) \in \mathbb{R}^3$.

$$(2+\lambda)x - 3\lambda x = 0 \tag{1}$$
$$3y-(2+\lambda)y = 0 \tag{2}$$
$$5+3\lambda z - 5\lambda - 3z = 0 \tag{3}$$

For (1),

$$(2+\lambda)x-3\lambda x = 0 \iff 2(1-\lambda)x = 0 \iff \lambda = 1,$$

because this condition is true for every $(x,y,z) \in \mathbb{R}^3$. Similarly, (2) gives

$$3y-(2+\lambda)y = 0 \iff (1-\lambda)y = 0 \iff \lambda = 1,$$

because this condition is true for every $(x,y,z) \in \mathbb{R}^3$. Finally, (3) yields

$$5+3\lambda z-5\lambda-3z = 0 \iff (1-\lambda)(5-3z) = 0 \iff \lambda = 1,$$

because this condition is true for every $(x,y,z) \in \mathbb{R}^3$.
We conclude that the vector field F derives from a potential if, and only if, $\lambda = 1$.

2. Following the previous point, necessarily $\lambda = 1$. We set $F = \mathrm{grad}\phi$ and resolve:

$$\begin{aligned} \partial_x\phi = x^2+5\lambda y+3yz &\overset{!}{=} F_1 = x^2+5\lambda y+3yz && \Longrightarrow \phi(x,y,z) = \frac{x^3}{3}+5yx+3xyz+\alpha_1(y,z) \\ \partial_y\phi = 5x+3xz+\partial_y\alpha_1(y,z) &\overset{!}{=} F_2 = 5x+3xz-2 && \Longrightarrow \phi(x,y,z) = \frac{x^3}{3}+5yx+3xyz-2y+\alpha_2(z) \\ \partial_z\phi = 3xy+\partial_z\alpha_2(z) &\overset{!}{=} F_3 = 3xy-4z && \Longrightarrow \phi(x,y,z) = \frac{x^3}{3}+5yx+3xyz-2y-2z^2+\alpha_3. \end{aligned}$$

The condition $\phi(3,2,-1) = 0$ implies

$$0 = \phi(3,2,-1) = 9+15-18-2+\alpha_3 \iff \alpha_3 = 4.$$

The sought potential is then:

$$\phi(x,y,z) = \frac{x^3}{3}+5yx+3xyz-2y-2z^2+4.$$

Common mistakes: part 1

- Forgetting to remark on the requirement for convex and/or simply connected domain.
- Mistakes in the sign while computing the curl.
- Adding together the three components instead of writing it as a vector (it was a system of 3 equations, not a single equation).

Common mistakes: part 2

- Mistakes in the integration constants like leaving out dependence on x, y, z, introducing multiple constants when only one is needed, or not explaining how they went from $\alpha(y, z)$, $\beta(x, z)$, $\gamma(x, y)$ to the final form of the potential.
- Minor mistakes in the integration for the $2y$, $2z^2$ terms.
- Arithmetic mistakes in the computation of the constant final constant for fixing $\phi(3, 1, -2) = 0$, or forgetting entirely to compute the constant.

Question 8.

We start computing $\iint_S \mathrm{curl} F \cdot \mathrm{d}s$.

A paramterization of the upper semi-sphere is just given by $\sigma(\theta,\varphi) = (3\cos\theta\sin\varphi, 3\sin\theta\sin\varphi, 3\cos\varphi)$ with $(\theta,\varphi) \in [0,2\pi]\times[0,\pi/2] =: A$. We compute

$$\sigma_\theta = (-3\sin\theta\sin\varphi, 3\cos\theta\sin\varphi, 0)$$
$$\sigma_\varphi = (3\cos\theta\cos\varphi, 3\sin\theta\cos\varphi, -3\sin\varphi),$$

and

$$\sigma_\theta \wedge \sigma_\varphi = \begin{pmatrix} -9\cos\theta\sin^2\varphi \\ -9\sin\theta\sin^2\varphi \\ -9\sin\varphi\cos\varphi \end{pmatrix}.$$

In addition, $\mathrm{curl}F = (0,0,-2)$. Thus

$$\begin{aligned}\iint_S \mathrm{curl}F \cdot \mathrm{d}s &= -2\int_0^{2\pi}\int_0^{\pi/2} -9\sin\varphi\cos\varphi \mathrm{d}\varphi\mathrm{d}\theta \\ &= 2\cdot 9\cdot 2\pi\cdot\frac{1}{2}\int_0^{\pi/2}\sin 2\varphi \mathrm{d}\varphi = 18\pi\left[-\frac{1}{2}\cos(2\varphi)\right]_0^{\pi/2} = 18\pi.\end{aligned}$$

Now we compute $\int_C F\cdot \mathrm{d}l$, with $C = \sigma(\partial A) = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$, where

$\Gamma_1 = \{\sigma(\theta,0) = (0,0,3) \mid \theta : 0 \to 2\pi\}$ $\quad$ $\Gamma_2 = \{\sigma(2\pi,\varphi) = (3\sin\varphi, 0, 3\cos\varphi) \mid \varphi : 0 \to \pi/2\}$

$\Gamma_3 = \{\sigma(\theta,\pi/2) = (3\cos\theta, 3\sin\theta, 0) \mid \theta : 2\pi \to 0\}$ $\quad$ $\Gamma_4 = \{\sigma(0,\varphi) = (3\sin\varphi, 0, 3\cos\varphi) \mid \varphi : \pi/2 \to 0\}$

It is clear that Γ_1 is a point and that the curves Γ_2 and Γ_4 have the same image but with opposite senses. Thus, the boundary positively orient of the semi-sphere is given by Γ_3, whose parameterization is obtained as $\gamma(t) = (3\cos(2\pi-t), 3\sin(2\pi-t), 0)$, $t\in[0,2\pi]$. In addition, $\gamma'(t) = (3\sin(2\pi - t), -3\cos(2\pi-t), 0)$. We have then:

$$\begin{aligned}\int_C F\cdot\mathrm{d}l &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t \\ &= \int_0^{2\pi} (-3\sin t, -3\cos t, 0)\cdot(-3\sin t, -3\cos t, 0)\,\mathrm{d}t \\ &= \int_0^{2\pi}\left(9\sin^2(t) + 9\cos^2(t)\right)\mathrm{d}t = 9\int_0^{2\pi}\mathrm{d}t \\ &= 18\pi\end{aligned}$$

Then, the equality is verified:

$$\iint_S \mathrm{curl}F\cdot\mathrm{d}s = \int_C F\cdot\mathrm{d}l.$$

Common mistakes

- $\text{curl}F$:
  - $\text{curl}F(x,y,z) = (0,0,\frac{d}{dx}F_y - \frac{d}{dy}F_x) = (0,0,-1-1) = (0,0,-2) \neq (0,0,2)$
  - $(\text{curl}F)(\sigma(\theta,\varphi)) = (0,0,-2) \neq (0,0,-6\cos\phi)$
    Warning: $\text{curl}F$ is the constant vector $(0,0,-2)$ for every $(x,y,z) \in \mathbb{R}^3$, *i.e.*, when we evaluate it at the point $\sigma(\theta,\varphi)$, it is equal to $(0,0,-2)$.

- Normal vector: during the course we defined

$$\int_S \text{curl}F \cdot ds = \int_A (\text{curl}F)(\sigma(\theta,\varphi)) \cdot (\sigma_\theta \wedge \sigma_\varphi)\, d\theta\, d\varphi\,,$$

  i.e., for computing the integral, it is not necessary to choose the outer normal vector, but simply $\sigma_\theta \wedge \sigma_\varphi$. If we use the parameterization given in the exercise, $\sigma_\theta \wedge \sigma_\varphi$ is an inner normal vector and it is not necessary to change its orientation. If we modify the parameterization of the exercise and consider, *e.g.*, $\sigma(\varphi,\theta)$ instead of $\sigma(\theta,\varphi)$, then $\sigma_\varphi \wedge \sigma_\theta$ is an outer normal vector.

- ∂S: with the exercise's paramterization, the parameterization of the boundary is given by $\gamma(t) = (3\cos t, -3\sin t, 0)$ and not $\tilde{\gamma}(t) = (3\cos t, 3\sin t, 0)$.

Question 9.

- The function is even thus $b_k = 0$ for $k \geq 1$.
- a_0

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+\cos(2x)+|x|)\mathrm{d}x$$
$$= 2 + \frac{2}{\pi}\int_0^{\pi}|x|\mathrm{d}x = 2+\pi$$

- $a_k,\ k \geq 1$

  The Fourier series is a linear operator; then

$$F(1+\cos(2x)+|x|) = F(1+\cos(2x)) + F(|x|).$$

  We denote by $\bar{a}_k$ the coefficients of $F(1+\cos(2x))$ and by $\tilde{a}_k$ the coefficients of $F(|x|)$. As $1+\cos(2x)$ is a trigonometric polynomial, it is straightforward to realize that $F(1+\cos(2x)) = 1+\cos(2x)$, and that $\bar{a}_2 = 1$ and $\bar{a}_k = 0$ for every $k \neq 0, 2$. We now compute $\tilde{a}_k$ $(k \neq 0)$:

$$\begin{aligned}\tilde{a}_k &= \frac{2}{2\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}x\cos(kx)\mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{1}{k}x\sin(kx)\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin(kx)\mathrm{d}x\right) \\ &= \frac{2}{\pi k^2}\Big[\cos(kx)\Big]_0^{\pi} = \frac{2}{\pi k^2}((-1)^k - 1)\end{aligned}$$

  Finally, $a_k = \bar{a}_k + \tilde{a}_k$ and we conclude:

$$\begin{cases} a_0 = 2+\pi & \\ a_2 = \bar{a}_2 + \tilde{a}_2 = 1 & \\ a_{2k} = \bar{a}_{2k} + \tilde{a}_{2k} = 0 & \text{for } k \geq 2 \\ a_{2k+1} = \bar{a}_{2k+1} + \tilde{a}_{2k+1} = -\frac{4}{\pi(2k+1)^2} & \text{for } k \geq 0 \end{cases}$$

Common mistakes

- Some coefficients (a_0 and a_2) must be treated separately,
- Use the following wrong statement "even function $\Rightarrow a_n = 0$".
- The domain in the exercise was $[-\pi, \pi]$ and not $[0, 2\pi]$. In particular the function $|x|$ over the domain $[-\pi, \pi]$ and extended by periodicity is not the same as $|x|$ over the domain $[0, 2\pi]$ and extended by periodicity.

Question 10.

1. The function $f(x)$ is 2π-periodic and piecewise regular. Using Parseval's theorem, and the identities provied in the question, we get:

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}(x^3)^2\mathrm{d}x = \sum_{n=1}^{\infty} b_n^2 &= \sum_{n=1}^{\infty} \frac{4\pi^4}{n^2} - \frac{48\pi^2}{n^4} + \frac{144}{n^6} \\
&= 4\pi^4\frac{\pi^2}{6} - 48\pi^2\frac{\pi^4}{90} + 144\sum_{n=1}^{\infty}\frac{1}{n^6} \\
&= \frac{2\pi^6}{15} + 144\sum_{n=1}^{\infty}\frac{1}{n^6}, \qquad (4)
\end{aligned}$$

On the other hand,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^6\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^6\mathrm{d}x = \frac{2}{7\pi}\pi^7 = \frac{2\pi^6}{7}. \qquad (5)$$

Finally, making (4) equal to (5), we obtain:

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

2. We set $h(x) = x^4/4$. It is clear that $Fh(x) = Fg(x)/4$ (this equality holds for the coefficients term by term). We denote a_k^h and b_k^h the Fourier coefficients of $Fh(x)$, a_k^f and b_k^f, a_k^g and b_k^g those of Ff and Fg, respectively. As h is even, $b_k^h = 0$ for every $k \geq 1$. In addition, for $k \geq 1$,

$$\begin{aligned}
a_k^h &= \frac{2}{2\pi}\int_{-\pi}^{\pi}\frac{x^4}{4}\cos(kx)\mathrm{d}x \\
&= \left[\frac{x^4}{4\pi k}\sin(kx)\right]_{-\pi}^{\pi} - \frac{1}{k\pi}\int_{-\pi}^{\pi} x^3\sin(kx)\mathrm{d}x \\
&= -\frac{1}{k}b_k^f
\end{aligned}$$

Finally, we directly compute a_0^g:

$$a_0^g = \frac{2}{2\pi}\int_{-\pi}^{\pi} x^4\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^4\mathrm{d}x = \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2\pi^4}{5}$$

We conclude

$$a_0^g = \frac{2\pi^4}{5}, \text{ and } \begin{cases} a_k^g = 4a_k^h & = -\frac{4}{k}b_k^f \\ b_k^g = 0 \end{cases} \text{ pour } k \geq 1$$

Common mistakes: part 1

- State Parseval's theorem with the integration bounds $(0, 2\pi)$ instead of $(-\pi, \pi)$.
- To forget the coefficient $\frac{2}{2T}$ in front of the integral.
- To forget the square of the function in Parseval's theorem.
- Incorrect computation of b_n^2.

Common mistakes: part 2

- Forgetting the coefficient 4 in the equality between the Fourier coefficients of $f$and g.
- Incorrect computation of a_0.

Question 11.

We reformule the equation

$$2u(x) + 3f \star u(x) = f(x),$$

with $f(x) = e^{-|x|}$. Applying the Fourier transform we obtain

$$2\hat{u}(\alpha) + \sqrt{2\pi} \cdot 3\hat{f}(\alpha)\hat{u}(\alpha) = \hat{f},$$

with the convention $\hat{g}(\alpha) = \mathcal{F}[g](\alpha)$. We isolate $\hat{u}$ and obtain:

$$\hat{u}(\alpha) = \frac{\hat{f}(\alpha)}{2 + 3\hat{f}(\alpha)\sqrt{2\pi}}.$$

In addition,

$$\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} e^{-i\alpha t} \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{(1-i\alpha)t} \mathrm{d}t + \int_{0}^{\infty} e^{-(1+i\alpha)t} \mathrm{d}t \right) \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{2}{1+\alpha^2}.
\end{aligned}$$

Thus:

$$\hat{u}(\alpha) = \frac{\frac{1}{\sqrt{2\pi}} \frac{2}{1+\alpha^2}}{2 + 3\frac{1}{\sqrt{2\pi}} \frac{2}{1+\alpha^2} \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}(4+\alpha^2)}.$$

Finally, we conclude

$$\begin{aligned}
u(x) &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \mathcal{F}^{-1} \left(\sqrt{\frac{2}{\pi}} \frac{1}{2^2 + \alpha^2} \right) \\
&= \frac{1}{2} \frac{e^{-|2x|}}{2} = \frac{e^{-2|x|}}{4}.
\end{aligned}$$

Common mistakes

- Most common error: at the end, after correctly identifying the table's line, to use $e^{-2|x|} = e^{-2x}$.
- To forget the factor $\sqrt{2\pi}$ when applying the Fourier transform to a convolution product $\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f) \cdot \mathcal{F}(g)$
- To forget the factor 3 in the equation when applying the Fourier transform.

- To not use the table of Fourier transform for turning back to the space domain (solution expressed in the Fourier transform form).
- Wrong use of the table. In particular, line 3 instead of line 7.
- After a precendent calculation mistake, it is possible to find a solution of $\hat{u}$ such that, its inverse Fourier transform is given by:

$$u(x) = \begin{cases} e^{-|wx|} & \text{if } x > 0 \quad (w > 0) \\ 0 & \text{otherwise} \end{cases}$$

  (Given by the table.) Frequently, the students concluded that $u(x) = e^{-|wx|}$ without specifiying the case $x < 0$.

- Common calculation mistakes:

$$2 + \frac{6}{1+\alpha^2} = \frac{2\alpha^2+7}{\alpha^2+1}$$
$$\frac{1}{2\alpha^2+8} = \frac{2}{\alpha^2+4}$$