Teacher : TEACHER NAME
EXAM NAME - MAN
DATE
Duration : XXX minutes

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Student One

SCIPER : **111111** Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 8 pages, the last ones possibly blank. Do not unstaple.

- Sign this first page.
- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam except for the provided colored scratch paper.
- The scratch will not be collected at the end of exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  +2 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The points for the **open questions** may be different from one to the other. Please refer to the marking scheme given at the top of each question.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Pour votre examen, imprimez de préférence les documents compilés à l'aide de auto-multiple-choice.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = \left(x^2 + y^2 + z^2, xy, z\right).$$

Then:

- [x] $\mathrm{div}(\mathrm{curl}(F)) = 0$ over $\mathbb{R}^3$
- [ ] $\mathrm{div}(F) = 0$ over $\mathbb{R}^3$
- [ ] $F = \nabla f$, for any $f \in C^1(\mathbb{R}^3)$
- [ ] $\mathrm{curl}(F) = 0$ over $\mathbb{R}^3$

Question 2 Let $T > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{T}{2}\right[\\ -1 & \text{if } x \in \left[\frac{T}{2}, T\right[\end{cases}$$

extended by T-periodicity to $\mathbb{R}$. Its Fourier series is:

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi\,(2n+1)} \sin\left(\frac{2\pi}{T}\,(2n+1)\,x\right).$$

The sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + n + 1/4}$ is equal to:

- [ ] 0
- [x] $\pi^2/2$
- [ ] $\pi/16$
- [ ] $\pi^2/8$

Question 3 Consider the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ defined such that:

$$g(x) = e^{-\frac{x^2}{2}}, \quad \text{and} \quad h(x) = g\left(\frac{x}{4}\right).$$

Then, the Fourier transform $\hat{h} = \mathcal{F}(h)$ verifies:

- [ ] $\hat{h}'(\alpha) = i\frac{\alpha}{4} e^{-\frac{\alpha^2}{32}}$
- [ ] $\displaystyle\int_{-\infty}^{\infty} |\hat{h}'(\alpha)| d\alpha = \sqrt{2\pi}$
- [ ] $\hat{h}(\alpha) = e^{-8\alpha^2}$
- [x] $\displaystyle\left(\frac{1}{2}\frac{d}{d\alpha}[e^{4\alpha^2}\hat{h}(\alpha)]\right)^2 = 64\alpha^2\hat{h}(\alpha)$

Question 4 Let $u : \mathbb{R} \to \mathbb{R}$ be a C^1-periodic solution of the following system:

$$\begin{cases} u'(x) + 3u(x) = \cos(3x) + \sin(5x), \quad \forall x \in \mathbb{R}, \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi). \end{cases}$$

Then:

- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{6}\cos(3x) - \frac{5}{34}\sin(5x) + \frac{3}{28}\cos(5x)$
- [x] $u(x) = \frac{1}{6}\sin(3x) + \frac{1}{6}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{5}{34}\cos(5x)$
- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{1}{8}\cos(3x) + \frac{3}{34}\sin(5x) - \frac{5}{34}\cos(5x)$
- [ ] $u(x) = \frac{1}{8}\sin(3x) + \frac{3}{28}\cos(5x)$

Question 5 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $F(x, y) = (y, -x)$, and let $Q \subset \mathbb{R}^2$ be a square whose four sides have length 2. We denoted by Γ the boundary of Q negatively oriented (or clockwise oriented).

The line integral $\displaystyle\int_\Gamma F \cdot dl$ is equal to:

- [ ] -4
- [ ] 2
- [x] 8
- [ ] 4

Second part, open questions

Answer in the empty space below. Your answers should be carefully justified, and all the steps of your argument should be discussed in detail. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ■6

Let $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by:

$$F(x,y) = \left(\frac{18x}{9x^2 + (y-1)^2},\ y + \frac{2y-2}{9x^2 + (y-1)^2} \right).$$

1. Determine the domain of definition Ω of F.
2. Does F derives from a potential? If true, find its potential φ such that $\varphi(0,0) = 0$.

Question 7: *This question is worth 8 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ■ 8

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (x, 0, z).$$

Verify the Divergence Theorem applied to F over the domain Ω given by:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } y < 0\}.$$

Question 8: *This question is worth 8 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ■ 8

Let be $a, b \in \mathbb{R}$, and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$F(x, y) = (ay, bx)\,.$$

Verify the Green Theorem applied to F over the domain given by:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - \frac{1}{2})^2 < 1 \text{ and } y > 0\}.$$

Question 9: *This question is worth 7 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ■ 7

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that:

$$\int_0^\infty |f(x)| + |f'(x)|\, dx < \infty.$$

1. Write the definition of the Fourier transforms $\mathcal{F}(f)$ and $\mathcal{F}(f')$.
2. Express $\mathcal{F}(f')$ in terms of $\mathcal{F}(f)$.
3. Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined by $h(x) = f'(x+1)$. Express $\mathcal{F}(h)$ in terms of $\mathcal{F}(f)$.
4. Deduce that if additionally $f'(x+1) = f(x), \forall x \in \mathbb{R}$, then $f(x) = 0, \forall x \in \mathbb{R}$.

Question 10: *This question is worth 7 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ■7

Write explicitly the 2π periodic solution $u : \mathbb{R} \to \mathbb{R}$ of the following differential system:

$$\begin{cases} u'''(x) + 8u''(x) + 5u(x) = \sin(x), \quad \forall x \in \mathbb{R}, \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi). \end{cases}$$

Question 11: *This question is worth 6 points.*

□0 □1 □2 □3 □4 □5 ■6

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{if } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

By using the properties of the Fourier transform, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the following equation:

$$y(x) + 2y''(x) + \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \left(y''''(t) - 4y''(t) + 4y(t)\right) e^{-\sqrt{2}|x-t|}\, dt = 5e^{-5x^2}, \quad \forall x \in \mathbb{R}.$$

If necessary, you may use the table of Fourier transforms given above.