+1/1/60+

Teach. Pablo Antolin
Analysis III - (n/a)
January 18, 2022
Duration: 2 hours

n/a

n/a

SCIPER : **999999**

Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 28 pages, the last ones possibly blank. Do not unstaple.

- Sign this first page.
- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam except for the provided colored scratch paper.
- The scratch will not be collected at the end of exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  +2 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The points for the **open questions** may be different from one to the other. Please refer to the marking scheme given at the top of each question.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Consider a curve $C \subset \mathbb{R}^2$ connecting the point $P = (1, 0)$ to the point $Q = (0, 1)$, and let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$G(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2} \right).$$

Then:

- ☐ $\int_C G \cdot dl = \ln(2)$
- ☐ $\int_C G \cdot dl = -\ln(2)$
- ☐ $\int_C G \cdot dl$ depends on the curve C
- ☐ $\int_C G \cdot dl = 0$

Question 2 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar field defined by:

$$f(x, y) = x^2 + y^3 + xy.$$

Its Laplacian $\Delta f = \operatorname{div}(\nabla f)$ at (x, y) is:

- ☐ $x + 3y^2 + x + y$
- ☐ $2 + 6y$
- ☐ 8
- ☐ $4 + 6y$

Question 3 : Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$F(x, y) = (x, y),$$

and let Γ be the curve defined by:

$$\Gamma = \left\{ (x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = R^2 \right\},$$

with $R \in \mathbb{R}, R > 0$. The line integral $\int_\Gamma F \cdot dl$ is equal to:

- ☐ $\frac{\pi}{2} R$
- ☐ $2\pi R$
- ☐ 0
- ☐ $4\pi R^2$

Question 4 : The non-zero complex Fourier coefficients of the function g defined by:

$$g(x) = \cos(x) + 3\sin(3x),$$

which enable to express g as:

$$g(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

are:

- ☐ c_1, c_{-1}, c_3, c_{-3}
- ☐ c_1, c_{-3}
- ☐ c_1, c_3
- ☐ $c_0, c_1, c_{-1}, c_3, c_{-3}$

Question 5 : Let $F : \Omega \to \mathbb{R}^2$, with $\Omega \subset \mathbb{R}^2$, be the vector field defined by:

$$F(x, y) = \left(x, 4\frac{x}{y}\right).$$

Does the field F derive from a potential?

- ☐ Yes, for any domain Ω.
- ☐ No.
- ☐ Yes, on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$.
- ☐ Yes, on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

Second part, open questions

Answer in the empty space below. Your answers should be carefully justified, and all the steps of your argument should be discussed in detail. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 *Do not write here.*

Let C be the closed curve parametrically described by:

$$C = \left\{ \gamma(t) = \left(\frac{1}{2} \sin 2t,\, \sin t \right) : t \in [0, \pi] \right\}.$$

1. Let also $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by:

$$F(x, y) = (-y, x).$$

   Compute the line integral:

$$I = \int_C F \cdot dl.$$

2. Let R denote the domain enclosed by the curve C. How does the area of R relate to the line integral I introduced in part 1?

Question 7: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Do not write here.*

Let be $\lambda \in \mathbb{R}$, with $\lambda > 0$.

1. For which value(s) of λ, does the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$F(x, y, z) = \left(x^2 + 5\lambda y + 3yz,\ 5x + 3\lambda xz - 2,\ (2 + \lambda)xy - 4z\right)$$

derive from a potential?

2. Consider the specific case $\lambda = 1$. Find the potential of F, denoted here φ, such that $\varphi(3, 1, -2) = 0$.

Question 8: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Do not write here.*

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (y, -x, 0)\,.$$

Verify Stokes' Theorem applied to F through the hemisphere:

$$S = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 9 \text{ and } z \geq 0\},$$

using a parameterization:

$$\sigma(\theta, \varphi) = (3\cos\theta\sin\varphi, 3\sin\theta\sin\varphi, 3\cos\varphi)\,, \quad \text{with} \quad (\theta, \varphi) \in \bar{A} = [0, 2\pi] \times [0, \pi/2].$$

The boundary of S is the circle $C = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 = 9 \text{ and } z = 0\}$.

Question 9: *This question is worth 7 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 *Do not write here.*

Determine the real Fourier coefficients of the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = 1 + \cos 2x + |x|, \text{ for } x \in [-\pi, \pi[\,,$$

and extended by 2π-periodicity.

Question 10: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 *Do not write here.*

The real Fourier coefficients of the function f defined by $f(x) = x^3$ on $[-\pi, \pi[$ and extended by 2π-periodicity are:

$$a_0 = 0, \quad a_n = 0, \ \forall n > 0, \quad b_n = -\frac{2\,(-1)^n\,\pi^2}{n} + \frac{12\,(-1)^n}{n^3}, \ \forall n > 0.$$

1. Knowing also that:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$
compute $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

2. Using the relation $\frac{1}{4}x^4 = \int x^3 dx$, determine the real Fourier coefficients of the function g defined by $g(x) = x^4$ on $[-\pi, \pi[$ and extended by 2π-periodicity.

Question 11: *This question is worth 8 points.*

0 1 2 3 4 5 6 7 8 *Do not write here.*

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{if } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert \alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

By using the table of Fourier transforms given above, find the function $u : \mathbb{R} \to \mathbb{R}$ solution of the following equation:

$$2u(x) + \int_{-\infty}^{+\infty} 3u(t)e^{-|x-t|}\, dt = e^{-|x|}, \quad \forall x \in \mathbb{R}.$$