+1/1/60+

1

Teacher : Pablo Antolin
Analysis III - Mock exam - Student
November 2024
Duration : 180 minutes

Student One

SCIPER : **111111**

Do not turn the page before the start of the exam. This document is double-sided, has 30 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  $+2$ points if your answer is correct,
  0 points if your answer is incorrect, you give no answer, or more than one answer is marked.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 The non-zero complex Fourier coefficients of the function $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x) = \cos(x) + 3\sin(3x),$$

which enables to express g as:

$$g(x) = \sum_{k=-\infty}^{\infty} = c_k e^{ikx}$$

are

- [x] c_1, c_{-1}, c_3, c_{-3}
- [ ] c_1, c_{-3}
- [ ] c_1, c_3
- [ ] c_0, c_1, c_{-1}, c_3, c_{-3}

Question 2 Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by:

$$F(x, y, z) = (x^2 + y^2 + z^2, xy, z).$$

Then:

- [x] $\mathrm{div}(\mathrm{curl}(F)) = 0$ over $\mathbb{R}^3$
- [ ] $\mathrm{div}(F) = 0$ over $\mathbb{R}^3$
- [ ] $F = \nabla f$, for any $f \in C^1(\mathbb{R}^3)$
- [ ] $\mathrm{curl}(F) = 0$ over $\mathbb{R}^3$

Question 3 Let F be the vector field defined by:

$$F : \mathbb{R}^2 \to \mathbb{R}^2;\ (x, y) \mapsto (x, y),$$

and let $R \in \mathbb{R}, R > 0$ and A be the domain defined by:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\right\}.$$

We also denote the boundary of A by ∂A, and the outer unit normal of ∂A by $\nu : \partial A \to \mathbb{R}^2$. The integral $\int_{\partial A} F \cdot \nu \, dl$ is equal to:

- [ ] 0
- [x] $2\pi R^2$
- [ ] $\frac{3}{4}\pi R$
- [ ] $\frac{3}{5}\pi^2 R$

Question 1:

$$g(x) = \cos(x) + 3\sin(3x)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$a_0 = 0,\ a_1 = 1,\ a_n = 0\ \forall n > 1$

$b_1 = 0,\ b_2 = 0,\ b_3 = 3,\ b_n = 0\ \forall n > 3.$

$c_n = \frac{a_n - ib_n}{2}$

$c_{-n} = \frac{a_n + ib_n}{2}$

$c_1 = \frac{a_1}{2} = 1/2 \neq 0$

$c_{-1} = \frac{a_1}{2} = 1/2 \neq 0$

$c_3 = \frac{-i3}{2} = -\frac{3}{2}i \neq 0,\quad c_{-3} = \frac{i3}{2} \neq 0$

Question 2: $F(x,y,z) = (x^2 + y^2 + z^2, xy, z)$

$\operatorname{div} F = 2x + x + 1 \neq 0$

$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial y}(x,y,z) - \frac{\partial F_2}{\partial z}(x,y,z), \frac{\partial F_1}{\partial z}(x,y,z) - \frac{\partial F_3}{\partial x}(x,y,z), \frac{\partial F_2}{\partial x}(x,y,z) - \frac{\partial F_1}{\partial y}(x,y,z) \right)$$

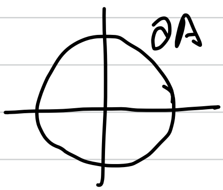
$\operatorname{curl} F = (0-0, 2z-0, y-2y) = (0, 2z, -y) \neq 0$ over $\mathbb{R}^3$.

As $\operatorname{curl} F \neq 0$ does not derive from a potential.

Then $F = \nabla f$ for $f \in C^1(\mathbb{R}^3)$ is not true.

Finally $\operatorname{div}(\operatorname{curl} F) = 0$

Question 3: Applying divergence theorem



$$\int_{\partial A} F \cdot \nu \, dl = \int_A \operatorname{div} F \, dA$$

$$F = (x, y) \quad , \quad \operatorname{div} F(x, y) = 2$$

$$\int_A \operatorname{div} F \, dA = 2 \int_A dA = 2\pi R^2$$

Question 4 Let F be the vector field defined by:

$$F:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2;\ (x,y)\mapsto\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right).$$

F is conservative (*i.e.* it derives from a potential)

- [x] over $\Omega=\{(x,y):\ 1\leq x\leq 3, 2\leq y\leq 10\}$.
- [ ] over $\Omega=\{(x,y):\ 2\leq x^2+y^2\leq 4\}$.
- [ ] over $\Omega=\{(x,y):\ x^2+y^2\leq 10\}$.
- [ ] for any domain Ω.

Question 5 Let $T>0$, and let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by:

$$f(x)=\begin{cases}1 & \text{if } x\in\left[0,\frac{T}{2}\right[\\ -1 & \text{if } x\in\left[\frac{T}{2},T\right[\end{cases}$$

extended by T-periodicity to $\mathbb{R}$. Its Fourier series is:

$$Ff(x)=\sum_{n=0}^{\infty}\frac{4}{\pi\,(2n+1)}\sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

The sum $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+n+1/4}$ is equal to:

- [ ] 0
- [x] $\pi^2/2$
- [ ] $\pi^2/16$
- [ ] $\pi^2/8$

Question 6 Let $u:\mathbb{R}\to\mathbb{R}$ be a C^1-periodic solution of the following system:

$$\begin{cases}u'(x)+3u(x)=\cos(3x)+\sin(5x), & \forall x\in\mathbb{R}\\ u(0)=u(2\pi)\\ u'(0)=u'(2\pi)\end{cases}$$

Then:

- [ ] $u(x)=\frac{1}{8}\sin(3x)+\frac{1}{6}\cos(3x)-\frac{5}{34}\sin(5x)+\frac{3}{28}\cos(5x)$
- [x] $u(x)=\frac{1}{6}\sin(3x)+\frac{1}{6}\cos(3x)+\frac{3}{34}\sin(5x)-\frac{5}{34}\cos(5x)$
- [ ] $u(x)=\frac{1}{8}\sin(3x)+\frac{1}{8}\cos(3x)+\frac{3}{34}\sin(5x)-\frac{3}{28}\cos(5x)$
- [ ] $u(x)=\frac{1}{8}\sin(3x)+\frac{3}{28}\cos(5x)$

Question 4:

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$$\text{curl}\, F(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)$$

$$= \frac{1}{(x^2+y^2)^2} \left(x^2+y^2-2x^2 + x^2+y^2 - 2y^2 \right)$$

$= 0$ (necessary condition) ✓

F is not defined at $(0,0)$.

☐ for any domain → is not defined at $(0,0)$ → False

☐ for $\Omega = \{ (x,y) : x^2+y^2 \leq 10 \}$

☐ for $\Omega = \{ (x,y) : 2 \leq x^2+y^2 \leq 4 \}$

↳ non-simply connected and non-convex

→ I don't know.

$\square$ for $\Omega = \{(x,y) : 1 \leq x \leq 3,\ 2 \leq y \leq 10\}$

F defined $\forall (x,y) \in \Omega$,

Ω is convex (and simply connected)

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

$\rightarrow$ F derives from a potential in Ω

Question 5:

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2+n+1/4} = \sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2}$$

$$Ff(x) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1/2} \sin\left(\frac{2\pi}{T}(2n+1)x\right)$$

1.3.2 Parseval identity • Theorem: let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic function such that f and f' are piecewise-defined. Then: $\frac{2}{T}\int_0^T [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$

$$\int_0^T [f(x)]^2 dx = \int_0^{T/2} (1)^2 dx + \int_{T/2}^{T} (-1)^2 dx = T$$

$$\frac{2}{T}\int_0^T [f(x)]^2 dx = \frac{2}{T} T = 2 = \frac{4}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(n+1/2)^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+n+1/4} = \frac{1}{2}\pi^2$$

Question 6: u is 2π-periodic (see boundary cond.)

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$u'(x) = \sum_{n=1}^{\infty} [-n a_n \sin nx + n b_n \cos nx]$$

$$u'(x) + 3u(x) = \cos(3x) + \sin(5x)$$

$$\frac{3}{2} a_0 + (n b_n + 3a_n)\cos nx + (3b_n - n a_n)\sin nx$$

$$= \cos 3x + \sin 5x$$

For $n=3$: $3b_3 + 3a_3 = 1$ and $3b_3 - 3a_3 = 0$

$\rightarrow a_3 = b_3 = 1/6$. You can find already the solution at this stage

For $n=5$: $5b_5 + 3a_5 = 0$ and $3b_5 - 5a_5 = 1$

$a_5 = -\frac{5}{3} b_5 \rightarrow 3b_5 + \frac{25}{3} b_5 = 1 \rightarrow b_5 = \frac{3}{34}$

$a_5 = -\frac{5}{34}$ ✓

Question 7 Let f be the scalar field defined by:

$$f : \mathbb{R}^2 \to \mathbb{R};\ (x, y) \mapsto xy + x + 1,$$

and let $R \in \mathbb{R}, R > 0$, and Γ the curved defined by:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = R^2\right\}.$$

The integral $\int_\Gamma f \,\mathrm{d}l$ is equal to:

- ☐ 0
- ☒ $2\pi R$
- ☐ $2\pi R^2 + \pi R + 1$
- ☐ $2\pi R^3 + \pi R^2 + R$

Question 8 Consider the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ defined such that:

$$g(x) = e^{-\frac{x^2}{2}}, \quad \text{and } h(x) = g\left(\frac{x}{4}\right).$$

Then, the Fourier transform $\hat{h} = \mathcal{F}(h)$ verifies:

- ☐ $\hat{h}'(\alpha) = i\frac{\alpha}{4}e^{-\frac{\alpha^2}{32}}$
- ☐ $\int_{-\infty}^{\infty} |\hat{h}'(\alpha)|\mathrm{d}\alpha = \sqrt{2\pi}$
- ☐ $\hat{h}(\alpha) = e^{-8\alpha^2}$
- ☒ $\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[e^{4\alpha^2}\hat{h}(\alpha)\right]\right)^2 = 64\alpha^2\hat{h}(\alpha)$

Question 7: $f(x,y) = xy + x + 1$

$$\int_\Gamma f \cdot dl \ , \quad \gamma(t) : [0, 2\pi] \to \mathbb{R}^2$$
$$\theta \mapsto (R\cos\theta, R\sin\theta)$$

$$\gamma'(t) = R(-\sin\theta, \cos\theta) \ , \ \|\gamma'(\theta)\| = R$$

$$\int_\Gamma f \, dl = \int_0^{2\pi} f(\gamma(\theta)) \, \|\gamma'(\theta)\| \, d\theta$$

$$= \int_0^{2\pi} (R^2 \cos\theta \sin\theta + R\cos\theta + 1) R \, d\theta$$

$$= R^3 \int_0^{2\pi} \cos\theta \sin\theta \, d\theta + R^2 \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{=0} + \underbrace{R \int_0^{2\pi} d\theta}_{2\pi R}$$

$$= 2\pi R + \frac{1}{2} R^3 \sin^2\theta \Big|_0^{2\pi} = 2\pi R$$

Question 8: $g(x) = e^{-\frac{x^2}{2}}$, $h(x) = g(\frac{x}{4})$

$h(x) = e^{-x^2/32}$

8	$f(y) = e^{-w^2 y^2}$	$(w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$

$\omega = \frac{1}{\sqrt{32}}$ " • $\hat{h}(\alpha) = \frac{\sqrt{32}}{\sqrt{2}} e^{-\frac{32}{4}\alpha^2} = 4e^{-8\alpha^2}$

~~□ $\hat{h}(\alpha) = e^{-8\alpha^2}$~~

• $\hat{h}'(\alpha) = -64\alpha e^{-8\alpha^2}$

~~□ $\hat{h}'(\alpha) = i\frac{\alpha}{4} e^{-\frac{\alpha^2}{32}}$~~

• $\frac{d}{d\alpha}\left(e^{4\alpha^2}\hat{h}(\alpha)\right) = \frac{d}{d\alpha}\left(4e^{-4\alpha^2}\right)$

$= -32\alpha e^{-4\alpha^2}$

$\left(\frac{1}{2}\frac{d}{d\alpha}\left(e^{4\alpha^2}\hat{h}(\alpha)\right)\right)^2 = \left(-16\alpha e^{-4\alpha^2}\right)$

$$= 16^2 \alpha^2 e^{-8\alpha^2} = 4\, e^{-8\alpha^2} 64\alpha^2 = 64\alpha^2 \hat{h}(\alpha)$$

☒ $$\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[e^{4\alpha^2}\hat{h}(\alpha)\right]\right)^2 = 64\alpha^2\hat{h}(\alpha)$$

- For the sake of completion, let's check

☐ $$\cancel{\int_{-\infty}^{\infty} |\hat{h}'(\alpha)|\mathrm{d}\alpha = \sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty} \left| -64\alpha e^{-8\alpha^2} \right| d\alpha =$$

$$64 \int_{-\infty}^{0} |\alpha| e^{-8\alpha^2} d\alpha + 64 \int_{0}^{\infty} |\alpha| e^{-8\alpha^2} d\alpha$$

$$= 64 \int_{-\infty}^{0} -\alpha e^{-8\alpha^2} d\alpha + 64 \int_{0}^{\infty} \alpha e^{-8\alpha^2} d\alpha$$

$$= 64 \int_{0}^{\infty} +\alpha e^{-8\alpha^2} d\alpha + 64 \int_{0}^{\infty} \alpha e^{-8\alpha^2} d\alpha$$

$$= 128 \int_{0}^{\infty} \alpha e^{-8\alpha^2} d\alpha = -\frac{128}{16} e^{-8\alpha^2}\Big|_{0}^{\infty} = 8 \neq 0$$

Second, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 8: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9

(i) Let Γ be the curve defined by

$$\Gamma = \left\{ \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}t^2}{2} \right) \,\middle|\, t \in [-1,1] \right\}.$$

Compute the length of Γ.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F(x,y) = \left(x^2, y\cos(x^2)\right)$$

and Ω the triangle whose vertices are $(0,0)$, $(\sqrt{\pi/2}, 0)$, and $(\sqrt{\pi/2}, \sqrt{\pi/2})$. Compute

$$\int_{\partial\Omega} F \cdot \nu \mathrm{d}l$$

where $\nu : \partial\Omega \to \mathbb{R}^2$ is outer unit normal field of the boundary of Ω.

i) $\gamma : [-1,1] \longrightarrow \mathbb{R}^3$

$t \longmapsto \left(\frac{1}{3}t^3, 3t, \frac{\sqrt{6}}{2}t^2\right)$

$\text{length}(\Gamma) = \int_\Gamma 1 \, dl = \int_{-1}^{1} \|\gamma'(t)\| \, dt$

$\gamma'(t) = (t^2, 3, \sqrt{6}\,t)$, $\|\gamma'(t)\| = \left(t^4 + 9 + 6t^2\right)^{1/2}$

$= t^2 + 3$

$\text{length}(\Gamma) = \int_{-1}^{1} (t^2+3)\, dt = \left[\frac{1}{3}t^3 + 3t\right]_{-1}^{1}$

$= \frac{2}{3} + 6 = \frac{20}{3}$

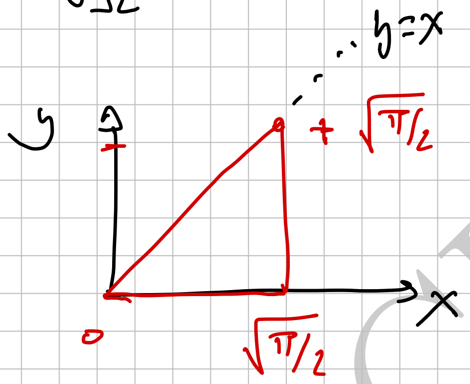
ii) $$\int_{\partial\Omega} F\cdot\nu \, dl = \int_{\Omega} \operatorname{div} F \, dA$$

div's theor.

$$F(x,y) = (x^2, y\cos(x^2)).$$

$$\operatorname{div} F(x,y) = 2x + \cos x^2$$

$$\int_{\Omega} \operatorname{div} F(x,y)\, dA = \int_{\Omega} (2x + \cos x^2)\, dx$$



$$\Omega = \{ (x,y) \in \mathbb{R}^2 : 0 \le y \le x \text{ and } 0 \le x \le \sqrt{\tfrac{\pi}{2}} \}$$

Then $$\int_{\Omega} (2x + \cos x^2)\, dx = \int_0^{\sqrt{\pi/2}} \int_0^{x} (2x + \cos x^2)\, dy\, dx$$

$$= \int_0^{\sqrt{\pi/2}} (2x^2 + x\cos x^2)\, dx$$

$$= \frac{2}{3}x^3 + \frac{1}{2}\sin x^2 \Big|_0^{\sqrt{\pi/2}} = \frac{2}{3}\left(\sqrt{\frac{\pi}{2}}\right)^3$$

$$+ \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi^{3/2}}{3\sqrt{2}} + \frac{1}{2}$$

Question 9: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right).$$

(i) Compute curl F.

(ii) Determine if F derives from a potential in Ω. If it does, find a potential of F, otherwise, justify why it does not derive from a potential in Ω.

i)

$$\operatorname{curl} F(x,y) = \frac{\partial F_y}{\partial x}(x,y) - \frac{\partial F_x}{\partial y}(x,y)$$

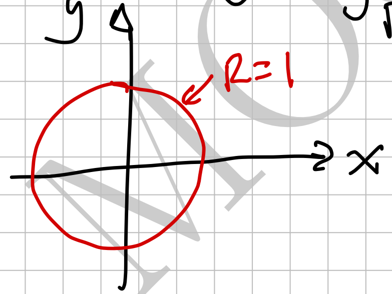
$$= \frac{1}{(x^2+y^2)^2}\left(x^2+y^2 - 2x(x+y) + x^2+y^2 + 2y(x-y) \right)$$

$$= 0$$

ii) Ω is not-simply connected or convex.

$\Rightarrow$ we don't know if F derives from a potential.

Let's try $\int_\Gamma F \cdot dl$ with



$\Gamma = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$\theta \longmapsto (\cos\theta, \sin\theta)$

$\gamma'(\theta) = (-\sin\theta, \cos\theta)$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(\theta)) \cdot \gamma'(\theta)\, d\theta =$$

$$= \int_0^{2\pi} (\cos\theta - \sin\theta, \cos\theta + \sin\theta) \cdot (-\sin\theta, \cos\theta)\, d\theta$$

$$= \int_0^{2\pi} -\cos\theta \sin\theta + \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta \, d\theta =$$

$$= \int_0^{2\pi} 1 \, d\theta = 2\pi \neq 0 \implies$$ F does not derive from a potential

Question 10: *This question is worth 3 points.*

0 1 2 3

Let $F : \mathbb{R}^2 \to \mathbb{R}^2; F(x, y) = (F_1(x, y), F_2(x, y))$, be a vector field such that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and div $F = 0$. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by:

$$G(x, y) = (F_2(-x, y), F_1(-x, y)) .$$

Show that G derives from a potential in $\mathbb{R}^2$.

The domain of G is $\mathbb{R}^2$ that is convex and simply connected. So, we just need to show that $\operatorname{curl} G(x,y) = 0$

$$\operatorname{curl} G(x,y) = \frac{\partial G_2}{\partial x}(x,y) - \frac{\partial G_1}{\partial y}(x,y)$$

$$= \frac{\partial F_1}{\partial x}(t,y) - \frac{\partial F_2}{\partial y}(t,y) \quad \text{with } t = -x$$

$$= \frac{\partial F_1}{\partial t}(t,y) \underbrace{\frac{\partial t}{\partial x}}_{=-1} - \frac{\partial F_2}{\partial y}(t,y)$$

$$= -\left(\frac{\partial F_1}{\partial t}(t,y) + \frac{\partial F_2}{\partial y}(t,y)\right)$$

$= -\operatorname{div} F(t,y) = 0 \Rightarrow$ G derives from a potential

Question 11: *This question is worth 14 points.*

0 1 2 3 4 5 6 7
8 9 10 11 12 13 14

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $F(x, y, z) = (0, x, 0)$ and let Σ be the surface defined by

$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \left(\sqrt{x^2 + y^2} + 1\right)\left(3 - \sqrt{x^2 + y^2}\right), y \geq 0, z \geq 0 \right\}.$$

Verify the Stokes theorem for F and Σ.
Note: if necessary, use the following formulas:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

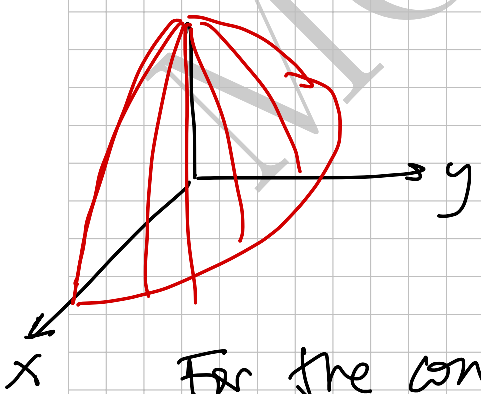
Let's first parameterize the surface Σ:

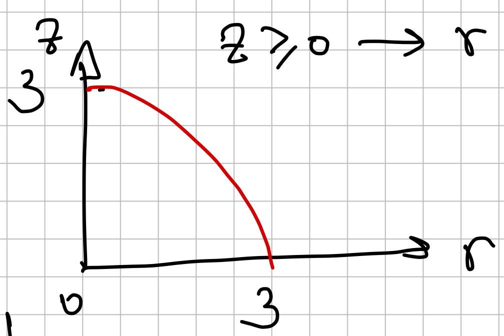
$\Sigma = \{ (x, y, z) \in \mathbb{R}^3 : z = (\underbrace{\sqrt{x^2+y^2}}_{r^2} + 1)(3 - \underbrace{\sqrt{x^2+y^2}}_{r^2}),$
$y \geq 0, z \geq 0 \}$

I select parametric coordinates (θ, r). s.t.

$x = r\cos\theta$, $y = r\sin\theta$, $\longrightarrow x^2 + y^2 = r^2$

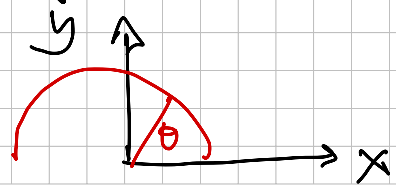
Then $z(r) = (r+1)(3-r)$





$z \geq 0 \longrightarrow r \in [0, 3]$

For the condition $y \geq 0 \longrightarrow \theta \in [0, \pi]$



Then, the parameterization is:

$$\sigma : A \longrightarrow \mathbb{R}^3$$
$$(r,\theta) \longmapsto \left(r\cos\theta,\ r\sin\theta,\ (r+1)(3-r)\right)$$

with $A = [0,3] \times [0,\pi]$

let's compute the normal

$$\sigma_r = \frac{\partial \sigma}{\partial r} = (\cos\theta, \sin\theta, 2-2r)$$

$$\sigma_\theta = \frac{\partial \sigma}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\sigma_r \times \sigma_\theta = r \begin{pmatrix} (2r-2)\cos\theta \\ (2r-2)\sin\theta \\ 1 \end{pmatrix}$$

<u>Important note</u>: as Σ is not a closed surface, it makes no sense talking about inner or outer normal. For verifying Stokes' theorem you just pick one parameterization (and stick to it) and compute the corresponding normal. No need to check the direction.

Stokes' theorem:

$$\iint_\Sigma \text{curl} F \cdot dS = \int_{\partial\Sigma} F \cdot dl$$

$$\text{curl} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Then, the left-hand-side of Stokes' theorem is:

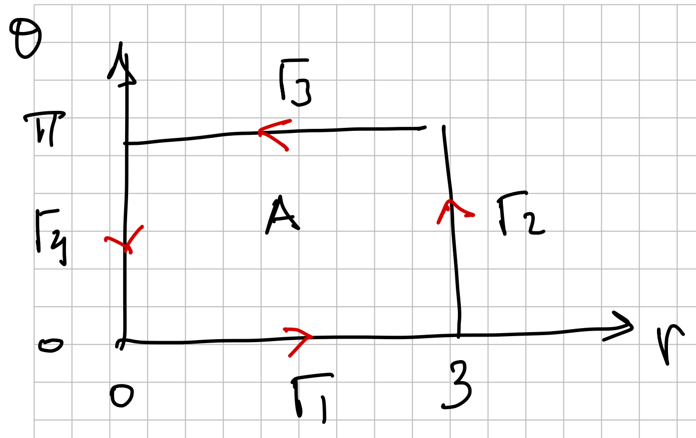
$$\iint_\Sigma \text{curl} F \cdot dS = \int_0^3 \int_0^\pi (\text{curl} F)(\sigma(r,\theta)) \cdot \sigma_r \times \sigma_\theta \, d\theta \, dr$$

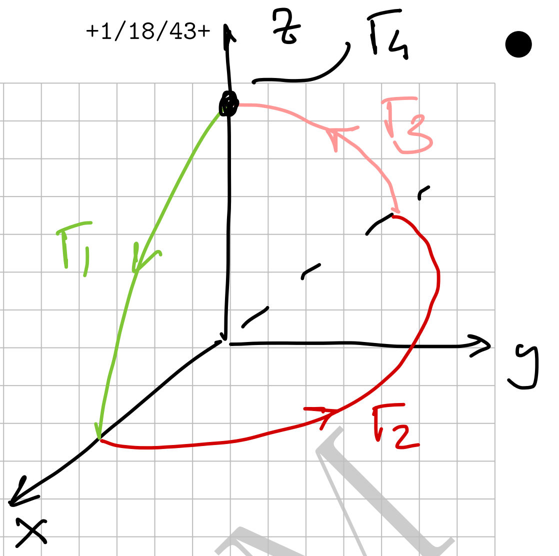
$$= \int_0^3 \int_0^\pi r(0,0,1) \cdot ((2r-2)\cos\theta, (2r-2)\sin\theta, 1) \, d\theta \, dr$$

$$= \int_0^3 \int_0^\pi r \, d\theta \, dr = \pi \int_0^3 r \, dr = \frac{\pi}{2} r^2 \Big|_0^3 = \frac{9}{2}\pi$$

- Now, the right-hand-side:

$\int_{\partial\Sigma} F \cdot dl$, but first we have to identify the boundary of Σ.





$\Gamma_1 = \{ \gamma_1(r) = \sigma(r,0) = (r, 0, (r+1)(3-r)) \text{ with } r: 0 \to 3 \}$

$\Gamma_2 = \{ \gamma_2(\theta) = \sigma(3,\theta) = (3\cos\theta, 3\sin\theta, 0) \text{ with } \theta: 0 \to \pi \}$

$\Gamma_3 = \{ \gamma_3(r) = \sigma(r,\pi) = (-r, 0, (r+1)(3-r)) \text{ with } r: 3 \to 0 \}$

$\Gamma_4 = \{ \gamma_4(\theta) : \sigma(0,\theta) = \underbrace{(0, 0, 3)}_{\text{single point}} \text{ with } \theta: \pi \to 0 \}$

Then $\partial\Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

$$\int_{\partial\Sigma} F \cdot d\ell = \int_{\Gamma_1} F \cdot d\ell + \int_{\Gamma_2} F \cdot d\ell + \int_{\Gamma_3} F \cdot d\ell$$

$$\int_{\Gamma_1} F \cdot d\ell = \int_0^3 \underbrace{(0, r, 0)}_{F(\gamma_1(r))} \cdot \underbrace{(1, 0, 2-r)}_{\gamma_1'(r)} dr = 0$$

$$\int_{\Gamma_2} F \cdot d\ell = \int_0^\pi \underbrace{3(0, \cos\theta, 0)}_{F(\gamma_2(\theta))} \cdot \underbrace{3(-\sin\theta, \cos\theta, 0)}_{\gamma_2'(\theta)} d\theta$$

$$= 9\int_0^{\pi} \cos^2\theta \, d\theta = \frac{9}{2}\int_0^{\pi} (1+\cos 2\theta)\, d\theta$$

$$= \frac{9\pi}{2} + \underbrace{\int_0^{\pi} \cos 2\theta \, d\theta}_{=0} = \frac{9\pi}{2}$$

$$\int_{\Gamma_3} F \cdot dl = \int_3^0 \underbrace{(0,-r,0)}_{F(\gamma_3(r))} \cdot \underbrace{(-1,\ 0,\ 2-r)}_{\gamma_3'(r)}\, dr = 0$$

Then: $\int_{\partial\Sigma} F \cdot dl = 9\pi/2$ that is equal to

$$\iint_{\Sigma} \operatorname{curl} F \cdot ds = 9\pi/2 .$$

Question 12: *This question is worth 9 points.*

0 1 2 3 4 5 6 7 8 9

Let $f : [0, \pi] \to \mathbb{R}$ be the function $f(x) = -x^2 + 2\pi x$.

(i) Compute $F_s f$, the Fourier series in sines of f.

(ii) Using the course's results, compare $F_s f$ and f in the interval $[0, \pi]$.

i) $L = \pi$

$$F_s f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right) = \sum_{n=1}^{\infty} b_n \sin nx$$

$L = \pi$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin nx = \frac{2}{\pi} \int_0^{\pi} (-x^2 + 2\pi x) \sin nx \, dx$$

$u = 2\pi x - x^2$

$dv = \sin nx \, dx$

$v = -\frac{1}{n} \cos nx$

$$= \frac{2}{\pi n} \Big[-(2\pi x - x^2) \cos nx \Big]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \frac{1}{n} (2\pi - 2x) \cos nx \, dx$$

$$= -\frac{2}{\pi n} \pi^2 (-1)^n + \frac{4}{\pi n} \int_0^{\pi} (\pi - x) \cos nx \, dx$$

$$= \frac{2\pi}{n} (-1)^{n+1} + \frac{4}{\pi n} \underbrace{\int_0^{\pi} \pi \cos nx \, dx}_{=0} - \frac{4}{\pi n} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2\pi}{n} (-1)^{n+1} - \frac{4}{\pi n} \int_0^{\pi} x \cos nx \, dx$$

$= \Theta$ +1/22/39+

$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n}\left[\frac{x}{n}\sin nx\right]_0^{\pi} + \frac{4}{\pi n}\int_0^{\pi}\frac{1}{n}\sin nx\,dx$$

$u = x$
$dv = \cos nx\,dx$
$v = +\frac{1}{n}\sin nx$

$$= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^2}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n^3}\left[(-1)^n - 1\right]$$

$$= \frac{2\pi}{n}(-1)^{n+1} - \frac{4}{\pi n^3}(-1)^n + \frac{4}{\pi n^3}$$

$$= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}(-1)^{n+1} + \frac{4}{\pi n^3}$$

$$= \left(\frac{2\pi}{n} + \frac{4}{\pi n^3}\right)(-1)^{n+1} + \frac{4}{\pi n^3}$$

Then:

$$Fs\,f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{\pi n^3} + \frac{2\pi^2 n^2 + 4}{\pi n^3}(-1)^{n+1}\right]\sin nx$$

ii) $f(x)$ is continuous in $[0, \pi]$

$f(0) = 0$ but $f(\pi) = \pi^2 \neq 0$.

Then, we know that

$$f(x) \neq F_s f(x) \quad \forall x \in \mathbb{R}.$$

Note: further conclusions can be drawn if $f(x)$ is extended to $[-\pi, 0]$ as $-f(-x)$, then $f(x)$ becoming an odd function in $[-\pi, \pi]$. Extending it by 2π-periodicity and computing its "standard" Fourier series, a similar expression as the one of Fourier-sine series would be obtained. Applying then the Dirichlet theorem, some conclusions can be drawn about the comparison of $f(x)$ and $F_s f(x)$

Question 13: *This question is worth 4 points.*

0 1 2 3 4

Let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x & \text{if} \quad -\pi \le x \le 0 \\ \pi & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{extended by } 2\pi\text{-periodicity.}$$

The real Fourier coefficients of g are

$$a_0 = \frac{3\pi}{2};$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases} \qquad \text{for } n \ge 1;$$

$$b_n = \frac{1}{n} \qquad \text{for } n \ge 1.$$

Using those coefficients and one result of the course, compute the sum

$$\sum_{k=1}^{+\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2}.$$

The Fourier series of $g(x)$ is:

$$Fg(x) = \frac{3\pi}{4} - \sum_{n \text{ odd}} \frac{2}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$= \frac{3\pi}{4} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Evaluating the series at $x=0$ we get

$$Fg(0) = \frac{3\pi}{4} - \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2\pi} \underbrace{\cos(n \cdot 0)}_{=1} + \sum_{n=1}^{\infty} \frac{1}{n} \underbrace{\sin(n \cdot 0)}_{=0}$$

$$= \frac{3\pi}{4} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$= \frac{3\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3\pi}{4} - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{(k-1/2)^2}$$

$k = n+1$

$n = k-1$

Now, applying Dirichlet theorem.

$$Fg(0) = \frac{1}{2}\left(g(0^+) + g(0^-)\right) = \frac{1}{2}(\pi + 0) = \frac{\pi}{2}$$

Then:

$$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(k-1/2)^2}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{(k-1/2)^2} = \frac{\pi^2}{2}$$

Question 14: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

(i) Write the definition of the Fourier transform of a function detailing its hypotheses

(ii) Using the properties of the Fourier transform, find $u : \mathbb{R} \to \mathbb{R}$, the solution of

$$-10u(x) + \int_{-\infty}^{+\infty} \Big(9u(t) - 4u''(t)\Big) e^{-\frac{3}{2}|x-t|} \mathrm{d}t = \frac{4x^2}{(2\pi + x^2)^2}.$$

If needed, use the Fourier transforms of the table below.

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{si } y > 0 \\ 0, & \text{sinon} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{si } b < y < c \\ 0, & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$

i) let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise-defined function s.t. $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$ then:

$$\hat{f} : \mathbb{R} \to \mathbb{C}$$
$$\alpha \mapsto \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

(ii)

$$-10\, u(x) + \int_{-\infty}^{+\infty} (9u(t) - 4u''(t))\, e^{-\frac{3}{2}|x-t|}\, dt$$

$$= \frac{4x^2}{(2\pi + x^2)^2}$$

We identify the convolution:

$$-10\, u(x) \quad (9u(x) - 4u''(x)) * e^{-\frac{3}{2}|x|} = g(x)$$

where $g(x) = \dfrac{4x^2}{(2\pi + x^2)^2}$

We apply the Fourier transform to both sides (checking in the given table is simple to realize that all the involved terms can be Fourier-transformed).

$$-10\, \hat{u}(\alpha) + \mathcal{F}\left((9u(x) - 4u''(x)) * e^{-\frac{3}{2}|x|}\right)(\alpha)$$

$$= \hat{g}(\alpha)$$

Using Fourier transform properties:

$$-10\, \hat{u}(\alpha) + \sqrt{2\pi}\, \mathcal{F}(9u(x) - 4u''(x))(\alpha)$$

$$\cdot\, \mathcal{F}(e^{-3/2|x|})(\alpha) = \hat{g}(\alpha)$$

and:

$$-10\,\hat{u}(\alpha) + \sqrt{2\pi}\left(9\,\hat{u}(\alpha) - 4(i\alpha)^2\hat{u}(\alpha)\right)$$

- $\mathcal{F}(e^{-3/2|x|})(\alpha) = \hat{g}(\alpha)$

then:

$$\mathcal{F}(e^{-3/2|x|})(\alpha) = \frac{3}{2}\sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha^2+9/4}$$

Table: row 7

Thus:

$$-10\,\hat{u}(\alpha) + \sqrt{2\pi}\,(9+4\alpha^2)\,\hat{u}(\alpha)\frac{3}{2}\sqrt{\frac{2}{\pi}}\,\frac{4}{9+4\alpha^2}$$

$= \hat{g}(\alpha)$ that becomes:

$$\hat{u}(\alpha)\left[12-10\right] = \hat{g}(\alpha) \Rightarrow \hat{u}(\alpha) = \frac{1}{2}\,\hat{g}(\alpha)$$

And applying the inverse Fourier transform

$$u(x) = \frac{1}{2}\,g(\alpha) = \frac{2x^2}{(2\pi+x^2)^2}$$