

Some indications for the solutions that are not immediate.

Exercise 1.

This is immediate.

Exercise 2.

This is immediate.

Exercise 3.

This is immediate.

Exercise 4.

This is immediate. See Exercise 5 on Shet 4.

Exercise 5. 1. Immediate.

2. Immediate.

3. Immediate.

4. The subgroup H is the kernel of the map

$$G \rightarrow (\mathbb{Z}/p)^\times, \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto a$$

which is a morphism by the first point. So we use the 1st isomorphism theorem.

5. If $F \cong \mathbb{Z}/p$ then F is cyclic. Assume that $F \neq H$. Then $F \cap H$ is trivial, as both groups have prime orders. As $F \subset N_G(H) = G$, we obtain that $|\langle H, F \rangle| = |H| \cdot |F| = p^2$. But $|G| = p(p-1)$ by the previous point, so this is a contradiction.

6. By classification, G is isomorphic to a cyclic group or to S_3 . Now notice that G is not abelian.

Exercise 6. 1. Let $H' := \langle (34), (56) \rangle$. Then $\langle (12) \rangle$ and H' normalize each other and have trivial intersection. Thus

$$\langle (12), H' \rangle \cong \langle (12) \rangle \times H' \cong \mathbb{Z}/2 \times H'$$

A similar argument shows that $H' \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

2. Direct calculations.

3. $F \subset N_G(H)$ and $H \cap F$ is trivial, so $\langle H, F \rangle \cong H \rtimes F$. In particular $|\langle H, F \rangle| = |H| \cdot |F| = 8 \cdot 3 = 24$.

Exercise 7. 1. We have $\text{Aut}(\mathbb{Z}/3) \cong \mathbb{Z}/2$. Suppose we have a morphism $f: D_{10} \rightarrow \mathbb{Z}/2$. The order of $f(\sigma)$ must be divisible by 5, so $f(\sigma) = 0$. Thus f factors uniquely through a morphism $f': D_{10}/\langle \sigma \rangle \cong \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$. There are exactly two endomorphisms of $\mathbb{Z}/2$, the identity and the trivial one. Moreover f' is trivial if and only if f is trivial. So there is at most one non-trivial morphism f . On the other hand, there is clearly one, given by

$$D_{10} \xrightarrow{\text{quotient}} D_{10}/\langle \sigma \rangle \cong \mathbb{Z}/2 \xrightarrow{\text{id}} \mathbb{Z}/2.$$

2. The subgroup $\mathbb{Z}/3 \times \{e\}$ of $\mathbb{Z}/3 \rtimes_\phi D_{10}$ is normal, and the quotient is isomorphic to D_{10} . Now $\langle \sigma \rangle$ is normal of order 5 in D_{10} . By the correspondence theorem, the preimage of this normal subgroup, is a normal subgroup $N \leq \mathbb{Z}/3 \rtimes D_{10}$ of order $5 \cdot 3 = 15$.