Recall:

- $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = \dim \operatorname{Ran}(A)$
- $\operatorname{rank}(A) = \dim \operatorname{Ran}(A) = \dim \operatorname{Col}(A) = \dim \operatorname{Row}(A)$
  $= \#$ indep rows of A (or $\tilde{A}$)
  $= \#$ indep. colums of A (or $\tilde{A}$)
- The Rank Theorem: $A \in \mathbb{R}^{m\times n}$ (i.e., $f_A : \mathbb{R}^n \to \mathbb{R}^m$)

$$n = \dim \operatorname{Ker}(A) + \operatorname{rank}(A)$$

- Change of basis

  If $\mathcal{B}, \mathcal{C}$ are two bases of V and $\dim V = n$:

  Then the matrix $P_{C\leftarrow B} := ([b_1]_C, \dots, [b_n]_C) \in \mathbb{R}^{n\times n}$

  is invertible and $P_{C\leftarrow B} [x]_B = [x]_C$ for all $x \in V$.

- If $V = \mathbb{R}^n$ this can also be computed as follows:

  $P_B := (b_1 \dots b_n)$, $P_C = (c_1, \dots, c_n)$

  $P_{C\leftarrow B} = P_C^{-1} \cdot P_B$

Rmk Intuition for the Rank Theorem

Let V, W be vector spaces, $\dim V = n$, $\dim W = m$

Let $T: V \to W$ be a linear map (or equivalently, by Theorem 4.23 : $\tilde{T}: \mathbb{R}^n \to \mathbb{R}^m$ given by $A \in \mathbb{R}^{m \times n}$.)

We know : $\operatorname{Ker}(T) \subseteq \mathbb{R}^n$ subspace

Let $B_{Ker} = \{b_1 \dots b_k\}$ be a basis of $\operatorname{Ker}(T)$ $(\dim \operatorname{Ker}(A) = k \leq n)$

$\Rightarrow b_1 \dots b_k \subset V$ independent

$\Rightarrow \exists\, b_{k+1}, \dots, b_n \subset V$ so that $B_V = \{b_1 \dots b_k, \dots, b_n\}$ is a basis of V.

Clearly : $T(b_1) = \dots = T(b_k) = 0$ since $b_1 \dots b_k \in \operatorname{Ker}(T)$.

Consider $\tilde{B} := \{T(b_{k+1}), \dots, T(b_n)\}$

We observe :

- $\# \tilde{B} = n - k \overset{\text{Thm 4.19}}{=} \dim \operatorname{Ran}(T)$
- $T(b_{k+1}), \dots, T(b_n)$ are independent

Proof : $0 = \lambda_1 T(b_{k+1}) + \dots + \lambda_{n-k} T(b_n) = T(\lambda_1 b_{k+1} + \dots + \lambda_{n-k} b_n)$

$\Rightarrow \lambda_1 b_{k+1} + \dots + \lambda_{n-k} b_n \in \operatorname{Ker}(T)$

$\Rightarrow$ either $\lambda_1 = \dots = \lambda_{n-k} = 0$

or $0 \neq \lambda_1 b_{k+1} + \dots + \lambda_{n-k} b_n \in \operatorname{Ker}(T) = \operatorname{Span}\{b_1 \dots b_k\}$ ↯ ($b_1 \dots b_k \dots b_n$ indep!)

$\Rightarrow \lambda_1 = \dots = \lambda_{n-k} = 0$.

$\Rightarrow \tilde{B}$ is a basis of $\operatorname{Ran}(T)$

So, schematically

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m = \operatorname{Span}\{T(b_{k+1}), \dots T(b_n)\}$$

$$\mathbb{R}^n = \operatorname{Span}\{\underbrace{b_1 \dots b_k}_{\operatorname{Ker}(T) \xrightarrow{T} 0}, \underbrace{b_{k+1} \dots b_n}_{\xrightarrow{T}}\}$$

<u>Ex.</u> $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with $\operatorname{Rank}(T) = 1$

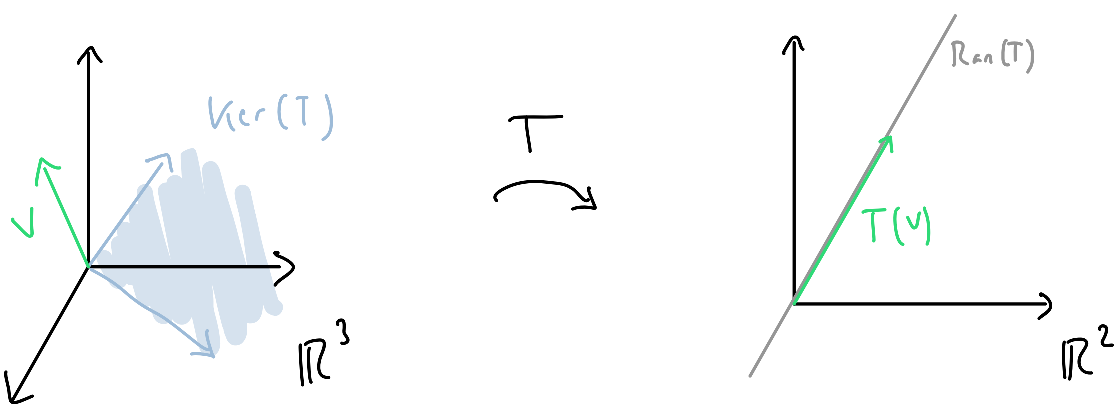
$\Rightarrow \dim \operatorname{Ker}(T) = 2 \quad \Rightarrow \operatorname{Ker}(T)$ is a plane in $\mathbb{R}^3$

Choose a vector v that is not in the plane

(e.g. v = normal vector for the plane)

$\Rightarrow \operatorname{Ran}(T) = \operatorname{Span}\{T(v)\}$ the line in $\mathbb{R}^2$

in direction $T(v)$



Expressing a map in terms of specific bases

Let V be a vector space and $B = \{b_1, \dots, b_n\}$ a basis of V,

and, W another vector space and $S = \{s_1, \dots, s_m\}$ a basis of W.

We know we can view V as $\mathbb{R}^n$ via the identification $v \mapsto [v]_B$

and, we can view W as $\mathbb{R}^m$ via the identification $w \mapsto [w]_S$

Now, given $T: V \to W$ linear, what does the corresponding map $\tilde{T}: \mathbb{R}^n \to \mathbb{R}^m$ look like? What is its matrix?

Recall: You saw a warm-up example for this in Week 8.

$$\left(V = \mathbb{P}_2,\ W = \mathbb{R}^2 \quad T(p) = \begin{pmatrix} p(0) \\ p'(1) \end{pmatrix} \right)$$

Thm. 4.23 Let V, W finite-dim vector spaces and $T: V \to W$ linear.

There exists a unique matrix $A \in \mathbb{R}^{m \times n}$ such that

$$[T(v)]_S = A\,[v]_B \qquad (*)$$

and this matrix A is given by:

$$A = \left([T(b_1)]_S, \dots, [T(b_n)]_S \right)$$

Def. We call A the matrix of T respective to the bases B and S.

Here is a schematic that explains, why (*) makes A the matrix that adequately represents T in the bases B and S:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\text{“translate” by } v \mapsto [v]_B"'] & W \arrow[d, "\text{“translate” by } w \mapsto [w]_S"] \\
\mathbb{R}^n \arrow[r, "\tilde{T}", "\text{given by } A\cdot"'] & \mathbb{R}^m
\end{tikzcd}$$

(*) $[T(v)]_S = A[v]_B$ means: the two paths have equal outcome

<u>Proof</u> Let $A := \left([T(b_1)]_S, \dots, [T(b_n)]_S \right)$

We need to prove (*).

Let $v \in V$ then $[v]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ for $\lambda_1 b_1 + \dots + \lambda_n b_n = v$

- $A[v]_B = \lambda_1 \cdot [T(b_1)]_S + \dots + \lambda_n [T(b_n)]_S$ (by def. of matrix multipl.)

- $[T(v)]_S = [T(\lambda_1 b_1 + \dots + \lambda_n b_n)]_S$

$= [\lambda_1 T(b_1) + \dots + \lambda_n T(b_n)]_S$ linearity of T

$= \lambda_1 \cdot [T(b_1)]_S + \dots + \lambda_n [T(b_n)]_S$ linearity of $x \mapsto [x]_S$ (Thm. 4.13)

same

$\square$

Ex. $T : \mathbb{P}_3 \to \mathbb{P}_2 \,, \quad T(p) = p'$

$$B = \{\underbrace{1}_{=b_1}, x, x^2, x^3\}, \quad S = \{1, x, x^2\}$$

Q: Give the matrix of the linear map T in terms of the bases B and S.

Solution:

- $T(b_1) = T(1) = 0$ (the zero polynomial in $\mathbb{P}_2$)

  $\Rightarrow [T(1)]_S = [0]_S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ because $0 = 0 \cdot \underbrace{1}_{=s_1} + 0 \cdot \underbrace{x}_{=s_2} + 0 \cdot \underbrace{x^2}_{=s_3}$

- $T(b_2) = T(x) = 1$

  $\Rightarrow [T(b_2)]_S = [1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ because $1 = 1 \cdot \underbrace{1}_{=s_1} + 0 \cdot \underbrace{x}_{=s_2} + 0 \cdot \underbrace{x^2}_{=s_3}$

- $T(b_3) = T(x^2) = 2x$

  $\Rightarrow [T(b_3)]_S = [2x] = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ because $2x = 0 \cdot \underbrace{1}_{=s_1} + 2 \cdot \underbrace{x}_{=s_2} + 0 \cdot \underbrace{x^2}_{=s_3}$

- $T(b_4) = T(x^3) = 3x^2$

  $\Rightarrow [T(b_3)]_S = [3x^2] = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ because $3x^2 = 0 \cdot \underbrace{1}_{=s_1} + 0 \cdot \underbrace{x}_{=s_2} + 3 \cdot \underbrace{x^2}_{=s_3}$

Hence: $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Here is an example in $V = \mathbb{R}^n$, $W = \mathbb{R}^m$:

<u>Ex.</u> $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x) = M(x)$ with $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad S = \left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Q : Give the matrix of the linear map f in terms of the bases B and S.

<u>Solution 1</u> : Analogous to previous example :

- $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left[f\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_S = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

- $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 1/2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left[f\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \\ 1/2 & 3/2 \end{pmatrix}$$

Solution 2 $\quad A = P_S^{-1} \cdot M \cdot P_B$

Schematic justification:

$$\begin{tikzcd}
{[v]_B} \arrow[r, "A"] \arrow[d, "P_B \cdot"'] & {[f(v)]_S} \\
v \arrow[r, "M \cdot"] & f(v) \arrow[u, "P_S^{-1} \cdot"']
\end{tikzcd}
\quad \Bigg\} \Rightarrow A = P_S^{-1} \cdot M \cdot P_B$$

Recall from week 8:

$$x = P_B \cdot [x]_B$$

$$P_B := (b_1 \dots b_n)$$

$$P_B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_S = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\Rightarrow P_S^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow A = P_S^{-1} \cdot M \cdot P_B$$

$$= \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \\ 1/2 & 3/2 \end{pmatrix}$$

$\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$ for $T: V \to W$

$T: V \to W$, $\dim V = n$, $\dim W = m$.

<u>Q</u>: How do we find (basis of) $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$ for a linear map $T: V \to W$? (In particular if $V \neq \mathbb{R}^n$ or $W \neq \mathbb{R}^m$)

Here is a good method: (See also the example at the bottom of page 10 in Week 8.)

1.) Choose bases for V and W:

$\mathcal{B} = \{b_1 \dots b_n\}$ for V

$S = \{s_1 \dots s_m\}$ for W

2.) Find the matrix of T respective to the bases $\mathcal{B}$ and S.

$$A = \left([T(b_1)]_S, \dots, [T(b_n)]_S \right)$$

3.) Compute $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ and their bases as in Example (✱) of Week 7.

4.) We can use identification of V with $\mathbb{R}^n$ and W with $\mathbb{R}^m$ to obtain bases for $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$

(This works thanks to Cor. 4.15.)

$\underline{\text{Ex}}$ $T : \mathbb{P}_3 \to \mathbb{P}_2$, $T(p) = p'$

Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$.

Choose the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ and the basis $S = \{1, x, x^2\}$ for $\mathbb{P}_2$ — 1. Step

From the previous example, we already know:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

— 2. Step

The columns $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ of A are independent and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in their Span. $\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$ is a basis of $\mathrm{Col}(A) = \mathrm{Ran}(A)$ — 3. Step

Translate $\mathbb{R}^3 \rightsquigarrow \mathbb{P}_2$: — 4. Step

$$[w_1]_S = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow w_1 = 1 \cdot \overbrace{s_1}^{=1} + 0 \cdot \overbrace{s_2}^{=x} + 0 \cdot \overbrace{s_3}^{=x^2} = 1$$

$$[w_2]_S = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \Rightarrow w_1 = 0 \cdot s_1 + 2 \cdot s_2 + 0 \cdot s_3 = 2x$$

$$[w_3]_S = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \Rightarrow w_1 = 0 \cdot s_1 + 0 \cdot s_2 + 3 \cdot s_3 = 3x^2$$

$\Rightarrow \{1, 2x, 3x^2\}$ is a basis of $\mathrm{Ran}(T)$.

(and $\mathrm{Ran}(T) = \mathrm{Span}\{1, 2x, 3x^2\} = \mathbb{P}_2$.)

Now the same for the Kernel: Step 3

Solve $Ax = 0$ as usual.

Here, A is already in RREF, so we easily find:

$$\mathrm{Ker}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\} \Rightarrow \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\} \text{ is a basis of } \mathrm{Ker}(A)$$

Translate $\mathbb{R}^4 \rightsquigarrow \mathbb{P}_3$. Step 4

$$[w]_B = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow w = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = 1 \in \mathbb{P}_3$$

$\Rightarrow \{1\}$ is a basis of $\mathrm{Ker}(T)$.

Def. Let V, W finite-dim vector spaces and $T: V \to W$ linear. Define $\mathrm{rank}(T) := \dim(\mathrm{Ran}(T))$

Rmk $\mathrm{rank}(T) = \mathrm{rank}(A)$ if A is the matrix of T respective to some bases B for V and S for W.

In the example above: $\mathrm{rank}(T) = \mathrm{rank}(A) = \dim \mathrm{Ran}(A) = 3$

5. Eigenvalues and eigenvectors

5.0 Complex numbers

Def. A complex number is a sum of the form $a+bi$ where $a,b \in \mathbb{R}$ and i is called the imaginary unit.

We say that two complex numbers $a+bi$ and $c+di$ are equal if $a=c$ and $b=d$.

We use the symbol $\mathbb{C}$ for the set of all complex numbers.

Def. We define addition and multiplication on $\mathbb{C}$ as follows:

$+$ in $\mathbb{R}$

- $(a+bi)+(c+di) := (a+c)+(b+d)i$
- $(a+bi)\cdot(c+di) := (ac-bd)+(bc+ad)i$

In conclusion, for $i = 0+1\cdot i$ we have $(i)^2 = (0\cdot 0-1\cdot 1)+(0\cdot 1-1\cdot 0)i = -1$.

Def. The real part and the imaginary part of $z = a+bi \in \mathbb{C}$ are defined by $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

The complex conjugate of z is $\bar{z} := a-bi$

and the absolute value of z is $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$.

Rmk. We interpret $\mathbb{R} \subseteq \mathbb{C}$ by $a = a + 0 \cdot i$

By easy but lengthy computations one can prove that for all $z, w, v \in \mathbb{C}$:

- $z + w = w + z$
- $z \cdot w = w \cdot z$
- $z + (w+v) = (z+w) + v$
- $z \cdot (w \cdot v) = (z \cdot w) \cdot v$
- $z + 0 = z$
- $z \cdot 1 = z$
- $z + \underbrace{(-1) \cdot z}_{=: -z} = 0$
- $z \cdot \underbrace{\frac{1}{|z|^2} \bar{z}}_{=: \frac{1}{z}} = 1$
- $z(w+v) = zw + zv$

(Since $\mathbb{C}$ with the operations $+$ and $\cdot$ satisfies the nine properties (axioms) above, we say that: $\mathbb{C}$ is a field.)

Further properties: For $z, w \in \mathbb{C}$

(a) $\bar{\bar{z}} = z$

(b) $\overline{z+w} = \bar{z} + \bar{w}$

(c) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(d) $|z \cdot w| = |z| \cdot |w|$

(Proof: easy, omitted)

Ex. $z = 1 + i$, $w = 3 - 2i$

Then:
- $z + w = (1+3) + (1-2)i = 4 - i$
- $z \cdot w = (3 \cdot 1 + 2 \cdot 1) + (-2 + 3)i = 5 + i$
- $\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot \frac{1}{|w|^2} \bar{w} = \frac{1}{13} z \cdot \bar{w} = \frac{1}{13}(1 + 5i) = \frac{1}{13} + \frac{5}{13} i$

$|w| = 3^2 + 2^2 = 13$ $\qquad$ $\bar{w} = 3 + 2i$

Geometric interpretation

$z = a+bi \quad , \quad w = c+di$

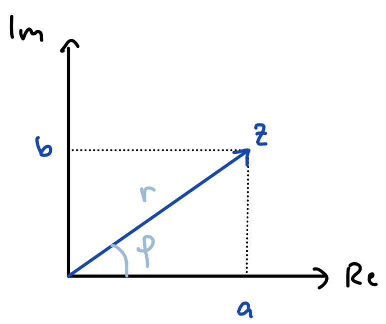
For addition $\mathbb{C}$ is exactly like $\mathbb{R}^2$ with vector addition:

$$z + w = (a+bi)+(c+di) = (a+c) + i\,(b+d)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

For multiplication, we still picture $\mathbb{C}$ to be the plane $\mathbb{R}^2$ but there is no analogy in vector operations.

In order to understand multiplication in $\mathbb{C}$, it is most convenient to express $z \in \mathbb{C}$ in polar coordinates:



$$z = a+bi$$
$$= r \cdot (\cos\varphi + i \sin(\varphi))$$

where $a = r \cdot \cos(\varphi)$
$\qquad b = r \sin(\varphi)$

or equivalently: $r = |z| = \sqrt{a^2+b^2}$ the radius of z

$\varphi = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$ the argument of z

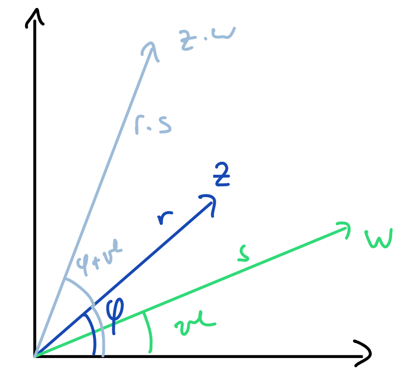
Multiplying two complex numbers is multiplying their radii and adding their arguments (angle with x-axis):

$z = r(\cos\varphi + i\sin\varphi)$, $w = s(\cos\vartheta + i\sin\vartheta)$

$$\Rightarrow z\cdot w = r\cdot s\big(\cos\varphi\cos\vartheta - \sin\varphi\sin\vartheta + i(\cos\varphi\sin\vartheta + \sin\varphi\cos\vartheta)\big)$$

$$= rs\big(\cos(\varphi+\vartheta) + i\sin(\varphi+\vartheta)\big)$$

angle sum identities for trigonometric fcts.



Thm. 4.23 (The fundamental Theorem of algebra)

Every polynomial with coefficients in $\mathbb{C}$ (or in $\mathbb{R} \subseteq \mathbb{C}$) of degree n has exactly n roots, counted with multiplicity.

(The proof is way beyond the scope of this course.)

Ex The polynomial $p(z) = z^4 + 6z^3 + 13z^2 + 24z + 36$

has coeff. in $\mathbb{R} \subseteq \mathbb{C}$ and $\deg = 4$

And indeed, we can factorize as follows

$$p(z) = (z - 2i)(z + 2i)(z - 3)^2$$

So $p(z)$ has: two roots of mult. $= 1$: $2i$ and $-2i$

and one root of mult $= 2$: 3

5.1. Eigenvalues and eigenvectors

Intuitively: an eigenvector for a matrix A is a vector $v \in \mathbb{R}^n$ that only changes in length (but not direction) under the linear map given by A, that is $Av = \lambda v$ for some scalar λ.

This condition only makes sense if also $Av \in \mathbb{R}^n$

$\rightsquigarrow$ so we only work with square matrices in this section!

Def. Let $A \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n \setminus \{0\}$, and λ a scalar in $\mathbb{R}$ (or $\mathbb{C}$).

If $Av = \lambda v$ then we call λ an eigenvalue of A and we call v an eigenvector of A for the eigenvalue λ.

For now: we think of λ as a real number. (More details later)

Ex. Is $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ an eigenvector of $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$?

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \underset{\substack{\uparrow \\ \text{want}}}{=} \lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} \text{There is no} \\ \lambda \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \text{such that } Av = \lambda v \end{array} \right.$$

$\Rightarrow$ v is not an eigenvector of A.

Ex. • $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ Q: Can you find / guess some eigenvalues and eigenvectors of A?

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \left\{ \begin{array}{l} v = \begin{pmatrix}1\\0\end{pmatrix} \text{ is an eigenvector for} \\ \text{the eigenvalue } \lambda = 2 \text{ of } A. \end{array}\right.$

$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -1\cdot\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \left\{ \begin{array}{l} v = \begin{pmatrix}0\\1\end{pmatrix} \text{ is an eigenvector for} \\ \text{the eigenvalue } \lambda = -1 \text{ of } A. \end{array}\right.$

• $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \left\{ \begin{array}{l} v = \begin{pmatrix}1\\1\end{pmatrix} \text{ is an eigenvector for} \\ \text{the eigenvalue } \lambda = 2 \text{ of } A. \end{array}\right.$

• $A = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \Rightarrow A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \left\{ \begin{array}{l} v = \begin{pmatrix}1\\0\end{pmatrix} \text{ is an eigenvector for} \\ \text{the eigenvalue } \lambda = 0 \text{ of } A. \end{array}\right.$

Q: Can a vector $v \neq 0$ $(v \in \mathbb{R}^n)$ be an eigenvector for two different eigenvalues λ and $\tilde{\lambda}$ of a matrix $A \in \mathbb{R}^{n\times n}$?

→ Let us check: Assume $\lambda v = Av = \tilde{\lambda} v \Rightarrow \lambda v = \tilde{\lambda} v$

Since $v \neq 0 \Rightarrow \lambda = \tilde{\lambda}$. So the answer is: no!

Q Can an eigenvalue $\lambda \in \mathbb{R}$ of $A \in \mathbb{R}^{n\times n}$ have two different eigenvectors v and $\tilde{v} \in \mathbb{R}^{n\times n}$?

Yes! e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Av = v = \underbrace{1}_{=\lambda}\cdot v \quad \forall v \in \mathbb{R}^n \Rightarrow \left\{ \begin{array}{l} \text{All } v \neq 0 \text{ are eigenvectors for} \\ \text{the eigenvalue } \lambda = 1 \text{ of } A. \end{array}\right.$

Q: How can we find out whether a given matrix has eigenvalues and eigenvectors?

$$Av \overset{?}{=} \lambda v \iff Av = \lambda \cdot I_n v$$
$$\iff \underbrace{(A - \lambda I_n)}_{\in \mathbb{R}^{n \times n}} v = 0$$

a linear system in matrix form!

- If for some λ the system $(A-\lambda I_n)v=0$ has a solution $v \neq 0$ then λ is an eigenvalue of A and v is an eigenvector of A for λ.

- If for some λ the system $(A-\lambda I_n)v=0$ has <u>no</u> solution $v \neq 0$ then λ is <u>not</u> an eigenvalue of A.

Observe: for $\lambda \in \mathbb{R}$,

$$(A-\lambda I_n)v = 0 \text{ for a } v \neq 0$$
$$\iff v \in \operatorname{Ker}(A-\lambda I_n) \text{ for a } v \neq 0$$
$$\iff \det(A-\lambda I_n) = 0.$$

So we have proven:

<u>Lemma 5.1</u>

λ is an eigenvalue of A $\iff$ $\det(A-\lambda I_n) = 0$.

Ex. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$$A - \lambda I_n = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$\Rightarrow \det(A - \lambda I_n) = (1-\lambda)(2-\lambda) - 3 \cdot 2$$
$$= \lambda^2 - 3\lambda + 2 - 6$$
$$= \lambda^2 - 3\lambda - 4$$
$$= (\lambda - 4)(\lambda + 1)$$

Hence $\det(A - \lambda I_n) = 0$ if and only if: $\lambda = -1$ or $\lambda = 4$

Lemma 5.1

$\Rightarrow$ the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ are -1 and 4.

What are the corresponding eigenvectors? $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$\rightarrow$ For both λs we have to find v so that $\underbrace{Av = \lambda v}_{\Leftrightarrow (A - \lambda I_n)v = 0}$

$\lambda = -1$

row reduction

$$A - \lambda I_n = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow (A - \lambda I_n)v = 0$ has solution space $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$\Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda = -1$

In fact every non-zero $w \in \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is an eigenvector for $\lambda = -1$.

$\lambda = 4$ $\quad A - \lambda I_n = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ $(A - \lambda I_n)v = 0$ has solution space $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$

$\Rightarrow$ $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector for $\lambda = 4$

In fact every non-zero $w \in \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$ is an eigenvector for $\lambda = 4$.

Summary of strategy:

1.) Find λs so that $\det(A - \lambda I_n) = 0$ $\rightsquigarrow$ eigenvalues λ

2.) For each λ, find solution space of $(A - \lambda I_n)v = 0$

$\rightsquigarrow$ (Space of) eigenvectors for each eigenvalue λ.

Why did we so easily find the λs for which $\det(A - \lambda_n) = 0$ in the previous example?

$\rightarrow$ $\det(A - \lambda I_n)$ was a polynomial $\quad$] true in general (see next Prop.)

$\rightarrow$ Polynomial had degree 2

$\Rightarrow$ We know exactly how to find roots $\quad$] only true if $A \in \mathbb{R}^{2 \times 2}$

Prop. 5.2 Let $A \in \mathbb{R}^{n\times n}$ and λ a scalar in $\mathbb{R}$ or $\mathbb{C}$.

Then the fct. $\chi_A(\lambda) := \det(\lambda I_n - A)$ is a polynomial of degree n with leading coefficient $= 1$.

i.e. we can write $\chi_A(\lambda) = \lambda^n + \sum_{k=0}^{n-1} b_k \lambda^k$ for some coefficients $b_k \in \mathbb{R}$.

Def. We call $\chi_A(\lambda)$ the characteristic polynomial of A.

Proof We prove the proposition by induction on n:

for $n=1$: $A = (a_{11}) \Rightarrow \lambda I_n - A = \lambda - a_{11}$

$\Rightarrow \det(\lambda I_n - A) = \lambda - a_{11}$

This is a polyn. of deg $= 1$ and leading coeff. $= 1$.

$(n-1) \rightsquigarrow n$: Now, we assume that the claim holds for $n-1$ and we prove it for n:

$$\lambda I_n - A = \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & & \vdots \\ \vdots & & \ddots & -a_{n-1\,n} \\ -a_{n1} & \cdots & -a_{n\,n-1} & \lambda - a_{nn} \end{pmatrix}$$

Laplace expansion by first row:

polyn. of deg = 1

$$\det(\lambda I_n - A) = (\lambda - a_{11}) \cdot \det((\lambda I_n - A)_{11})$$

$$+ \; a_{12} \cdot \det((\lambda I_n - A)_{12})$$

polyn. of deg = 0

$$\vdots$$

$$+ (-1)^{n+1} a_{1n} \cdot \det((\lambda I_n - A)_{1n})$$

By assumption, all the $\det((\lambda I_n - A)_{ii})$ are polynomials of degree $= n-1$ and leading coeff. $= 1$.

$\Rightarrow$ So each line in the above sum is a polynomial of degree n (first line) or $n-1$ (other lines), and each of them has leading coeff. $= 1$.

$\Rightarrow$ $\det(\lambda I_n - A)$ is a polyn. of degree $\leq n$.

Moreover, the first line is the only line that can have a term of degree n. So obviously that term is $1 \cdot \lambda^n$.

$\Rightarrow$ $\det(\lambda I_n - A)$ is a polyn. of degree $= n$ and leading coeff. $= 1$. $\square$

<u>Thm. 5.3</u> Let $A \in \mathbb{R}^{n\times n}$. A scalar λ in $\mathbb{R}$ or $\mathbb{C}$ is an eigenvalue of A if and only if $\chi_A(\lambda) = 0$. Also:

(i) A can have at most n distinct eigenvalues

(ii) A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

(iii) If A is a triangular matrix then its eigenvalues are its diagonal entries.

<u>Proof</u> The first part was shown in Lemma 5.1.

(i) By the fund. thm. of algebra $\chi_A(\lambda)$ can have at n roots (counted with multiplicity)

So, A can have at most n distinct eigenvalues

(ii) A invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow \det(-A) \neq 0$

$\Leftrightarrow \det(0 \cdot I_n - A) \neq 0$

$\Leftrightarrow \lambda = 0$ is not an eigenvalue of A

(iii) If A is triangular then $\lambda I_n - A$ is triangular

So $\det(\lambda I_n - A)$ is the product of its diagonal entries.

The diagonal entries of $\lambda I_n - A$ are: $\lambda - a_{11}, \dots, \lambda - a_{nn}$

$\Rightarrow \chi_A(\lambda) = \det(\lambda I_n - A) = (\lambda - a_{11}) \cdot \ldots \cdot (\lambda - a_{nn})$

$\Rightarrow$ the roots of the polyn. $\chi_A(\lambda)$ are: $a_{11}, a_{22}, \dots a_{nn}$

$\Rightarrow$ the eigenvalues of A are $a_{11}, a_{22}, \dots a_{nn}$ $\square$